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Problem 10 is out of order, see page 6

(a) $(1,-1,2)\cdot(0,2,-1)=-4$,
$(1,-1,2)\cdot(-1,1,1)=0$,
$(0,2,-1)\cdot(-1,1,1)=1$.
Thus the set is not orthogonal

(b) $(1,2,-1,1)\cdot(0,-1,-2,0)=0$
$(1,2,-1,1)\cdot(1,0,0,-1)=0$
$(0,-1,-2,0)\cdot(1,0,0,-1)=0$
Thus the set is orthogonal.

(c) $(0,1,0,-1)\cdot(1,0,1,1)=-1$
$(0,1,0,-1)\cdot(-1,1,-1,2)=-1$
$(1,0,1,1)\cdot(-1,1,-1,2)=0$
Thus the set is not orthogonal.

3. $\mathbf{u}\cdot\mathbf{v}=a-1-4=0$ only if $a=5$.

5. Let $\mathbf{u}_1=(1,-1,0)$ and $\mathbf{u}_2=(2,0,1)$. Define $\mathbf{v}_1=\mathbf{u}_1$. Compute

$$\mathbf{v}_2=\mathbf{u}_2-\left(\frac{\mathbf{u}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1=(2,0,1)-\tfrac{2}{2}(1,-1,0)=(1,1,1).$$

Then $\{\mathbf{v}_1,\mathbf{v}_2\}$ is an orthogonal basis for W. Let

$$\mathbf{w}_1=\frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1=\frac{1}{\sqrt{2}}(1,-1,0)\quad\text{and}\quad\mathbf{w}_2=\frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2=\frac{1}{\sqrt{3}}(1,1,1).$$

Then $\{\mathbf{w}_1,\mathbf{w}_2\}$ is an orthonormal basis for W.

7. Let $\mathbf{u}_1=(1,-1,0,1)$, $\mathbf{u}_2=(2,0,0,-1)$, and $\mathbf{u}_3=(0,0,1,0)$. Define $\mathbf{v}_1=\mathbf{u}_1$. Compute

$$\mathbf{v}_2=\mathbf{u}_2-\left(\frac{\mathbf{u}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1=(2,0,0,-1)-\tfrac{1}{3}(1,-1,0,1)=\left(\tfrac{5}{3},\tfrac{1}{3},0,-\tfrac{4}{3}\right),$$

$$\begin{aligned}\mathbf{v}_3&=\mathbf{u}_3-\left(\frac{\mathbf{u}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1-\left(\frac{\mathbf{u}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\right)\mathbf{v}_2\\&=(0,0,1,0)-0(1,-1,0,1)-0\left(\tfrac{5}{3},\tfrac{1}{3},0,-\tfrac{4}{3}\right)=(0,0,1,0).\end{aligned}$$

Then $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is an orthogonal basis for W. Clearing fractions in $\mathbf{v}_2$, we find that

$$\{\mathbf{v}_1',\mathbf{v}_2',\mathbf{v}_3'\}=\{(1,-1,0,1),(5,1,0,-4),(0,0,1,0)\}$$

is also an orthogonal basis for W. Let

$$\begin{aligned}\mathbf{w}_1&=\frac{1}{\|\mathbf{v}_1'\|}\mathbf{v}_1'=\frac{1}{\sqrt{3}}(1,-1,0,1),\\\mathbf{w}_2&=\frac{1}{\|\mathbf{v}_2'\|}\mathbf{v}_2'=\frac{1}{\sqrt{42}}(5,1,0,-4),\\\mathbf{w}_3&=\frac{1}{\|\mathbf{v}_3'\|}\mathbf{v}_3'=(0,0,1,0).\end{aligned}$$

Then $\{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3\}$ is an orthonormal basis for W.

9. Let $\mathbf{u}_1=(1,2)$ and $\mathbf{u}_2=(-3,4)$.

(a) Define $\mathbf{v}_1=\mathbf{u}_1$. Compute

$$\mathbf{v}_2=\mathbf{u}_2-\left(\frac{\mathbf{u}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1=(-3,4)-1(1,2)=(-4,2).$$

Then $\{\mathbf{v}_1,\mathbf{v}_2\}$ is an orthogonal basis for R^2.

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(b) Let

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{5}}(1,2) \quad \text{and} \quad \mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{2\sqrt{5}}(-4,2).$$

then $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthonormal basis for R^2.

11. Let $\mathbf{u}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$ and $\mathbf{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$. We first find a basis for R^3 containing $\mathbf{u}_1$ and $\mathbf{u}_2$ following the technique of Example 9 of Section 6.4. Then we use the Gram–Schmidt process to transform it to an orthonormal basis. Let

$$\mathbf{e}_1 = (1,0,0), \quad \mathbf{e}_2 = (0,1,0), \quad \mathbf{e}_3 = (0,0,1) \quad \text{and} \quad S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

S spans R^3 since it contains the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Form the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{e}_1 + c_4\mathbf{e}_2 + c_5\mathbf{e}_3 = (0,0,0)$$

which leads to the homogeneous system with augmented matrix

$$\left[\begin{array}{ccccc:c} \frac{2}{3} & \frac{2}{3} & 1 & 0 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 1 & 0 & 0 \\ \frac{1}{3} & -\frac{2}{3} & 0 & 0 & 1 & 0 \end{array}\right].$$

Transforming this augmented matrix to reduced row echelon form we obtain

$$\left[\begin{array}{ccccc:c} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 \end{array}\right].$$

The leading ones indicate that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{e}_1\}$ is a basis for R^3.

Next we use the Gram–Schmidt process. Let $\mathbf{v}_1 = \mathbf{u}_1$. Compute

$$\begin{aligned} \mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \mathbf{u}_2 - 0\mathbf{u}_1 = \left(\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right) \\ \mathbf{v}_3 &= \mathbf{e}_1 - \left(\frac{\mathbf{e}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{e}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\ &= \mathbf{e}_1 - \tfrac{2}{3}\mathbf{v}_1 - \tfrac{2}{3}\mathbf{v}_2 = \left(\tfrac{1}{9}, \tfrac{2}{9}, \tfrac{2}{9}\right). \end{aligned}$$

The set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for R^3. Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis, where

$$\begin{aligned} \mathbf{w}_1 &= \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \left(\tfrac{2}{3}, -\tfrac{2}{3}, \tfrac{1}{3}\right), \\ \mathbf{w}_2 &= \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \left(\tfrac{2}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\right), \\ \mathbf{w}_3 &= \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right). \end{aligned}$$

13. Let

$$\mathbf{u}_1 = (1,1,0,0), \quad \mathbf{u}_2 = (2,-1,0,1), \quad \mathbf{u}_3 = (3,-3,0,-2), \quad \mathbf{u}_4 = (1,-2,0,-3),$$

and

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}.$$

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(13 continued)

We find a basis for $W = \text{span } S$ following the method in Example 5 of Section 6.4. To determine a basis for span S form the linear combination

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4 = \mathbf{0}.$$

Expanding, adding vectors, and equating corresponding components from each side of the equation we obtain a homogeneous system with coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & -3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 \end{bmatrix}$$

which has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The leading 1's are in columns 1, 2, and 3 so $S_1 = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for span S. Next we use the Gram-Schmidt process. Let $\mathbf{v}_1 = \mathbf{u}_1$. Compute

$$\begin{aligned}
\mathbf{v}_2 &= \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = (2, -1, 0, 1) - \tfrac{1}{2}(1, 1, 0, 0) = \left(\tfrac{3}{2}, -\tfrac{3}{2}, 0, 1\right) \\
\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&= (3, -3, 0, -2) - 0(1, 1, 0, 0) - \tfrac{14}{11}\left(\tfrac{3}{2}, -\tfrac{3}{2}, 0, 1\right) = \left(\tfrac{12}{11}, -\tfrac{12}{11}, 0, -\tfrac{36}{11}\right).
\end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W. Clearing fractions, we have

$$\{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\} = \{(1, 1, 0, 0), (3, -3, 0, 2), (12, -12, 0, -36)\}$$

is also an orthogonal basis for W. Let

$$\begin{aligned}
\mathbf{w}_1 &= \frac{1}{\|\mathbf{v}_1'\|}\mathbf{v}_1' = \frac{1}{\sqrt{2}}(1, 1, 0, 0), \\
\mathbf{w}_2 &= \frac{1}{\|\mathbf{v}_2'\|}\mathbf{v}_2' = \frac{1}{\sqrt{22}}(3, -3, 0, 2), \\
\mathbf{w}_3 &= \frac{1}{\|\mathbf{v}_3'\|}\mathbf{v}_3' = \frac{1}{\sqrt{11}}(1, -1, 0, -3).
\end{aligned}$$

It follows that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis for W. 17

15. Let W be the subspace of R^4 consisting of all vectors of the form $(a, a+b, c, b+c)$. Since

$$(a, a+b, c, b+c) = a(1, 1, 0, 0) + b(0, 1, 0, 1) + c(0, 0, 1, 1)$$

it follows that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1, 1, 0, 0), (0, 1, 0, 1), (0, 0, 1, 1)\}$ spans W. To show that S is a basis for W we show that S is linearly independent. Form the expression

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = (0, 0, 0, 0)$$

which has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

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The reduced row echelon form of this matrix is

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_1 = c_2 = c_3 = 0$. Hence S is linearly independent. We next apply the Gram-Schmidt process to S. Let $\mathbf{v}_1 = \mathbf{u}_1$ and

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = (0,1,0,1) - \tfrac{1}{2}(1,1,0,0) = \left(-\tfrac{1}{2}, \tfrac{1}{2}, 0, 1\right)$$

$$\begin{aligned}\mathbf{v}_3 &= \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\ &= (0,0,1,1) - 0(1,1,0,0) - \left(\frac{1}{\frac{3}{2}}\right)\left(-\tfrac{1}{2}, \tfrac{1}{2}, 0, 1\right) = \left(\tfrac{1}{3}, -\tfrac{1}{3}, 1, \tfrac{1}{3}\right)\end{aligned}$$

Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis for W, where

$$\begin{aligned}\mathbf{w}_1 &= \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right) \\ \mathbf{w}_2 &= \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}\right) \\ \mathbf{w}_3 &= \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \left(\frac{1}{\sqrt{12}}, -\frac{1}{\sqrt{12}}, \frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}\right)\end{aligned}$$

17. Let W be the subspace of R^4 consisting of all vectors of the form (a, b, c, d) such that $a - b - 2c + d = 0$. We have $a = b + 2c - d$ so W is all vectors of the form

$$(b + 2c - d, b, c, d) = b(1,1,0,0) + c(2,0,1,0) + d(-1,0,0,1).$$

It follows that

$$S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{(1,1,0,0), (2,0,1,0), (-1,0,0,1)\}$$

spans W. To show that S is a basis for W we show that S is linearly independent. Form the expression

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = (0,0,0,0)$$

which has augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

The reduced row echelon form of this matrix is

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $c_1 = c_2 = c_3 = 0$. Hence S is linearly independent. We next apply the Gram-Schmidt process to S. Let $\mathbf{v}_1 = \mathbf{u}_1$ and

§ 6.8 (16) find orthonormal basis for subspace of $\mathbb{R}^3$ consisting of $\{(a,b,c): a+b+c=0\}$

Let W = subspace of $\mathbb{R}^3$ consisting of all (a,b,c) st $a+b+c=0$.

Notice that $a+b+c=0 \Rightarrow b=-a-c$

$\Rightarrow (a,b,c) = (a,-a-c,c) = a(1,0,-1) + c(0,-1,1)$

- So $S = \{(1,0,-1),(0,-1,1)\}$ spans W

To show S a basis for W, show S LI

Consider $c_1(1,0,-1) + c_2(0,-1,1) = (0,0,0)$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow c_1=c_2=0$$

- $\Rightarrow S$ is LI (S spans W + LI $\Rightarrow S$ a basis for W)

· Use Gram-Schmidt to orthonormalize S

$$v_1 = (1,0,-1) \;;\quad w_1 = \frac{v_1}{\|v_1\|} = \frac{(1,0,-1)}{\sqrt{1^2+0^2+(-1)^2}} = \underline{\left(\frac{1}{\sqrt2}, 0, \frac{-1}{\sqrt2}\right)}$$

$$v_2 = (0,-1,1) - \frac{(0,-1,1)(1,0,-1)}{(1,0,-1)(1,0,-1)}(1,0,-1)$$

$$= (0,-1,1) - \frac{-1}{2}(1,0,-1)$$

$$= \left(\tfrac12,-1,\tfrac12\right) \;;\quad w_2 = \frac{v_2}{\|v_2\|} = \frac{\left(\frac12,-1,\frac12\right)}{\sqrt{\left(\frac12\right)^2+1^2+\left(\frac12\right)^2}} = \frac{\left(\frac12,-1,\frac12\right)}{\sqrt{\frac64}}$$

$$= \underline{\left(\frac{1}{\sqrt6}, \frac{-2}{\sqrt6}, \frac{1}{\sqrt6}\right)}$$

- $\Rightarrow \boxed{\text{an orthonormal basis for } W = \left\{\left(1/\sqrt2, 0, -1/\sqrt2\right), \left(1/\sqrt6, -2/\sqrt6, 1/\sqrt6\right)\right\}}$

§ 6.8 (10) $u_1 = (1,1,1)$ $u_2 = (0,1,1)$ $u_3 = (1,2,3)$

(a) $v_1 = u_1$ $\quad |v_1| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$; $w_1 = \frac{v_1}{|v_1|} = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right)$

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1 = (0,1,1) - \left(\frac{0+1+1}{1+1+1}\right)(1,1,1) = (0,1,1) - \left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

$$= \left(-\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right);\ w_2 = \frac{v_2}{|v_2|} = \frac{\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)}{\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}} = \left(\frac{-2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right)$$

$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{u_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2 = (1,2,3) - \left(\frac{1+2+3}{1+1+1}\right)(1,1,1) - \left(\frac{-2+2+3}{4+1+1}\right)(-2,1,1)$$

$$= (1,2,3) - (2,2,2) - \left(-1, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right);\ w_3 = \frac{v_3}{|v_3|} = \frac{\left(0, -\frac{1}{2}, \frac{1}{2}\right)}{\sqrt{0+\frac{1}{4}+\frac{1}{4}}} = \left(0, \frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)$$

$$\Rightarrow \boxed{\{w_1, w_2, w_3\} = \left\{\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right), \left(\frac{-2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right), \left(0, \frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)\right\}}$$

(b) $(2,3,1) = a\cdot\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) + b\left(\frac{-2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right) + c\left(0, \frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

by Th 6.17 $\quad a = (2,3,1)\cdot\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) = \frac{6}{\sqrt3}$

$b = (2,3,1)\cdot\left(\frac{-2}{\sqrt6}, \frac{1}{\sqrt6}, \frac{1}{\sqrt6}\right) = 0$

$c = (2,3,1)\cdot\left(0, \frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right) = \frac{-2}{\sqrt2}$

$$\Rightarrow \boxed{(2,3,1) = \frac{6}{\sqrt3}\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right) - \frac{2}{\sqrt2}\left(0, \frac{-1}{\sqrt2}, \frac{1}{\sqrt2}\right)}$$

§ 6.8 (18) find orthonormal basis for sol'n space of

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right]$$

$$\Rightarrow \left.\begin{aligned} x_1 \quad + 3x_3 &= 0 \\ x_2 - 4x_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -3x_3 = -3s \\ x_2 &= 4x_3 = 4s \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3s \\ 4s \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \qquad \text{sol'n space basis} = \left\{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}\right\}$$

but need orthonormal basis vector

so $\dfrac{(-3,4,1)}{\sqrt{3^2+4^2+1^2}} = \dfrac{(-3,4,1)}{\sqrt{26}}$

$$\boxed{\text{so orthonormal basis for sol'n space is } \left\{\begin{bmatrix} -3/\sqrt{26} \\ 4/\sqrt{26} \\ 1/\sqrt{26} \end{bmatrix}\right\}}.$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = (2,0,1,0) - \tfrac{2}{2}(1,1,0,0) = (1,-1,1,0)$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$
$$= (-1,0,0,1) - \left(-\tfrac{1}{2}\right)(1,1,0,0) - \left(-\tfrac{1}{3}\right)(-1,0,0,1) = \left(-\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{3}, 1\right)$$

Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthonormal basis for W, where

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$
$$\mathbf{w}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)$$
$$\mathbf{w}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \left(-\frac{1}{\sqrt{42}}, \frac{1}{\sqrt{42}}, \frac{2}{\sqrt{42}}, \frac{6}{\sqrt{42}}\right)$$

19. Form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 2 & -6 & 0 \end{array}\right]$$

and find its reduced row echelon form. We obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and it follows that the solution is

$$\begin{bmatrix} -4t \\ 5t \\ t \end{bmatrix}$$

for any real number t. Hence

$$\mathbf{u}_1 = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

is a basis for the solution space. To find an orthonormal basis we compute

$$\frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \frac{1}{\sqrt{42}}\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}.$$

21. Let

$$\mathbf{w}_1 = \frac{1}{\sqrt{5}}(1,0,2), \quad \mathbf{w}_2 = \frac{1}{\sqrt{5}}(-2,0,1), \quad \mathbf{w}_3 = (0,1,0), \quad \text{and} \quad \mathbf{v} = (2,-3,1).$$

Following the procedure in Example 5, we compute

$$c_1 = \mathbf{v} \cdot \mathbf{w}_1 = \frac{4}{\sqrt{5}}, \quad c_2 = \mathbf{v} \cdot \mathbf{w}_2 = -\frac{3}{\sqrt{5}}, \quad c_3 = \mathbf{v} \cdot \mathbf{w}_3 = -3.$$

Then

$$\mathbf{v} = \frac{4}{\sqrt{5}}\mathbf{w}_1 - \frac{3}{\sqrt{5}}\mathbf{w}_2 - 3\mathbf{w}_3.$$

§ 6.8 20] $S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\}$ O.N basis of $\mathbb{R}^2$.

$$(2,3) = a\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + b\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Then by Thm. 6.17:

$$a = (2,3)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{5}{\sqrt{2}}.$$

$$b = (2,3)\cdot\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

$$(2,3) = \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

T.1. $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$ and 1 for $i = j$.

T.3. Since an orthonormal set of vectors is an orthogonal set, we know by Theorem 6.16 they are linearly independent. Since there are n of them, they span R^n.

T.5. If $\mathbf{u}$ is orthogonal to $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then $\mathbf{u} \cdot \mathbf{v}_j = 0$ for $j = 1, \ldots, n$. Let $\mathbf{w}$ be in span S. Then $\mathbf{w}$ is a linear combination of the vectors in S:

$$\mathbf{w} = \sum_{j=1}^{n} c_j \mathbf{v}_j.$$

Thus

$$\mathbf{u} \cdot \mathbf{w} = \sum_{j=1}^{n} c_j(\mathbf{u} \cdot \mathbf{v}_j) = \sum_{j=1}^{n} c_j 0 = 0.$$

Hence $\mathbf{u}$ is orthogonal to every vector in span S.

T.7. If $\mathbf{u} \cdot \mathbf{v} = 0$, then $u_1v_1 + u_2v_2 + \cdots + u_nv_n = 0$. We have

$$\mathbf{u} \cdot (c\mathbf{v}) = u_1(cv_1) + u_2(cv_2) + \cdots + u_n(cv_n) = c(u_1v_1 + u_2v_2 + \cdots + u_nv_n) = c(0) = 0.$$

T.9. Since some of the vectors $\mathbf{v}_j$ can be zero, A can be singular.

T.11 Let $\mathbf{x}$ be in S. Then we can write $\mathbf{x} = \sum_{j=1}^{k} c_j \mathbf{u}_j$. Similarly if $\mathbf{y}$ is in T, we have $\mathbf{y} = \sum_{i=k+1}^{n} c_i \mathbf{u}_i$. Then

$$\mathbf{x} \cdot \mathbf{y} = \left(\sum_{j=1}^{k} c_j \mathbf{u}_j\right) \cdot \mathbf{y} = \sum_{j=1}^{k} c_j(\mathbf{u}_j \cdot \mathbf{y}) = \sum_{j=1}^{k} c_j \left(\mathbf{u}_j \cdot \sum_{i=k+1}^{n} c_i \mathbf{u}_i\right) = \sum_{j=1}^{k} c_j \left(\sum_{i=k+1}^{n} c_i(\mathbf{u}_j \cdot \mathbf{u}_i)\right).$$

Since $j \neq i$, $\mathbf{u}_j \cdot \mathbf{u}_i = 0$, hence $\mathbf{x} \cdot \mathbf{y} = 0$.

ML.1. Use the following MATLAB commands.

```
A = [1 1 0;1 0 1;0 0 1];
gschmidt(A)

ans =

    0.7071    0.7071         0
    0.7071   -0.7071         0
         0         0    1.0000
```

Write the columns in terms of $\sqrt{2}$. Note that $\frac{\sqrt{2}}{2} \approx 0.7071$.

ML.3. To find the orthonormal basis we proceed as follows in MATLAB.

```
A = [0 -1 1;0 1 1;1 1 1]';
G = gschmidt(A)

G =

         0         0    1.0000
   -0.7071    0.7071         0
    0.7071    0.7071         0
```

To find the coordinates of each vector with respect to the orthonormal basis T which consists of the columns of matrix G we express each vector as a linear combination of the columns of G. It follows

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