(10) 1. $L: \mathbb{R}^2 \to \mathbb{R}^3$. Note $L(\underset{\sim}{x}) = Ax$. We need to find a the null space: $\{\underset{\sim}{x} \mid A\underset{\sim}{x} = 0\}$, where: $L(\underset{\sim}{x}) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Notice $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3$. $\qquad A \quad \underset{\sim}{x} = \underset{\sim}{0}$

(5) Now we solve:

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 \quad &= 0 \quad \Rightarrow x_1 = 0 = x_2. \\ -x_1 + x_2 &= 0 \end{aligned}$$

Therefore, for any $\underset{\sim}{x} \in \text{null}(A)$, $\underset{\sim}{x} = a\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for any $a \in \mathbb{R}$.
$\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ spans null(A). Linear independence is trivial.

(5) Thus basis of null(A) $= \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ or $\vec{0}$ (20)

2. $S = \{v_1, \dots, v_n\}$ is a basis $\Rightarrow S$ is linearly independent. (10)
Then A is row equivalent to I_n. $\Rightarrow \det A \neq 0$ (10)
Therefore A is non-singular and thus invertible.

3. A invertible so the only solution to $A\underset{\sim}{x} = \underset{\sim}{0}$ is $\underset{\sim}{x} = \underset{\sim}{0}$.
Consider: $c_1 Av_1 + \dots + c_n Av_n = 0$ where $c_1, \dots, c_n \in \mathbb{R}$.
Then: $A(c_1v_1 + \dots + c_nv_n) = 0 \Rightarrow c_1v_1 + \dots + c_nv_n = 0$
Since $S = \{v_1, \dots, v_n\}$ is a basis (by linear independence):
$c_1v_1 + \dots + c_nv_n = 0 \Leftrightarrow c_1 = \dots = c_n = 0$.
Thus: $c_1 Av_1 + \dots + c_n Av_n = 0 \Leftrightarrow c_1 = \dots = c_n = 0$

Hence $\{Av_1, \dots, Av_n\}$ is linearly independent.
That it also spans then follows since $\dim\{Av_1, \dots, Av_n\} = n$
Thus $\{Av_1, \dots, Av_n\}$ is basis for $\mathbb{R}^n$.

4) Consider: $c_1(3t+1) + c_2(3t^2+1) + c_3(2t^2+t+1) = 0$

Equating corresponding coefficients:

$$\begin{aligned} 2c_3 + 3c_2 \quad &= 0 \\ c_3 \quad + 3c_1 &= 0 \\ c_3 + \quad c_2 + c_1 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_3 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then: $$\mathrm{RREF}\left(\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]\right) = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $\{(3t+1), (3t^2+1), (2t^2+t+1)\}$ is not linearly independent and thus not a basis.

E.g. let $c_1 = 1 \Rightarrow c_2 = 2c_1 = 2$ and $c_3 = -3c_1 = -3$

whereas $(3t+1) + 2(3t^2+1) - 3(2t^2+t+1) = 6t^2 - 6t^2 + 3t - 3t + 3 - 3$
$= 0.$

-10 minimum for $3t+1$, $3t^2+1$, $2t^2+t+1$ are linearly independent.

5. $S_1 = \{v_1, v_2, v_3\}$ or $S_2 = \{v_1, v_2, v_3, v_4\}$

$\Rightarrow S_1 = \begin{bmatrix} 1 & 3 & 11 \\ 2 & 2 & 10 \\ 2 & 1 & 7 \end{bmatrix}$ or $S_2 = \begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix}$

$$\Rightarrow RREF(S_1) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & \\ v_1 & v_2 & \end{matrix} \end{array}$$

$\Rightarrow$ columns with leading 1's are 1 and 2 so $\{v_1, v_2\}$ is a basis for $W = \text{span } S_1$

$\dim W = 2$

$$\Rightarrow RREF(S_2) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \\ v_1 & v_2 & & \end{matrix} \end{array}$$

$\Rightarrow \{v_1, v_2\}$ is a basis for $W = \text{span } S_2$

$\dim W = 2$.

- either way will result in $\dim W = 2$. and $\text{span}(S_1) = \text{span}(S_2) = W$.
- Note $\{v_1, v_2\}$ is not the only basis, any two would suffice out of S

6\. Pick any of the vectors: $\{(1,1,1), (0,1,1), (1,2,3)\}$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_1 \qquad u_2 \qquad u_3$

let $v_1 = u_1 = (1,1,1)$

then $v_2 = u_2 - \dfrac{(u_2 \cdot v_1)}{(v_1 \cdot v_1)} v_1 = (0,1,1) - \frac{2}{3}(1,1,1)$

$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

$v_3 = u_3 - \dfrac{(u_3 \cdot v_1)}{(v_1 \cdot v_1)} v_1 - \dfrac{(u_3 \cdot v_2)}{(v_2 \cdot v_2)} v_2$

$= (1,2,3) - \frac{6}{3}(1,1,1) - \frac{3}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

(5) Then $\{v_1, v_2, v_3\} = \left\{(1,1,1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(0, -\frac{1}{2}, \frac{1}{2}\right)\right\}$

form an orthogonal basis.

−2 Small arithmetic error

−5 minimum for incorrect G-S formulation.