Name: Solution
Student ID #:

Mini-Quiz # 6

MAT-022A-Summer Session II (8/14/09)

You have until Monday 8/17/09 by class time. You may only use a pencil (or pen) and scrap paper. You must show all your work carefully.

1. Given a careful algebraic proof of the following statements. (2 points each).

   (a) Let x_1 and x_2 be solution to the linear system $Ax = 0$. Show that any linear combination of the solution vectors x_1, x_2 is a solution for $Ax = 0$.

   So we know that $Ax_1 = 0$ and $Ax_2 = 0$.
   Let $a\vec{x}_1 + b\vec{x}_2$ be a linear combination of x_1, x_2.
   Now $A(a\vec{x}_1 + b\vec{x}_2) = A(a\vec{x}_1) + A(b\vec{x}_2) = a(A\vec{x}_1) + b(A\vec{x}_2)$
   $= a(\vec{0}) + b(\vec{0}) = \vec{0} + \vec{0} = \vec{0}$.
   Thus $a\vec{x}_1 + b\vec{x}_2$ is a solution for $A\vec{x} = \vec{0}$.

   (b) Let A be a nonsingular $n \times n$ matrix and B be a singular $n \times n$ matrix. Show that AB is singular.

   $\det(A) \neq 0$ since A is nonsingular
   and
   $\det(B) = 0$ since B is singular.
   Now $\det(AB) = \det(A)\det(B) = 0$
   So AB is singular.

(c) Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be given. Show that $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ be given in $\mathbb{R}^n$.

Now $(u+v) \cdot w = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = (u_1+v_1) \cdot w_1 + \dots + (u_n+v_n) \cdot w_n$

$= u_1 w_1 + v_1 w_1 + \dots + u_n w_n + v_n w_n$

$= (u_1 w_1 + \dots + u_n w_n) + (v_1 w_1 + \dots + v_n w_n) = u_1 \cdot w + v \cdot w$

(d) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.

Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$(c\vec{u}) \cdot \vec{v} = \left(c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\right) \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = (cu_1)v_1 + \dots + (cu_n)v_n = c(u_1 v_1) + \dots + c(u_n v_n)$

$= c(u_1 v_1 + \dots + u_n v_n)$

$= c(\vec{u} \cdot \vec{v})$.

Also $(cu_1)v_1 + \dots + (cu_n)v_n = u_1(cv_1) + \dots + u_n(cv_n) = u \cdot (c\vec{v})$.

(e) Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that $L(\mathbf{0}_{\mathbb{R}^n}) = \mathbf{0}_{\mathbb{R}^m}$.

Now $L(\vec{0}_{\mathbb{R}^n}) = L(0\vec{0}_{\mathbb{R}^n}) \overset{\text{use linearity}}{=} 0\,L(0_{\mathbb{R}^n}) = \vec{0}_{\mathbb{R}^m}$.

Alternatively $L(0_{\mathbb{R}^n}) = L(0_{\mathbb{R}^n} + 0_{\mathbb{R}^n}) = L(0_{\mathbb{R}^n}) + L(0_{\mathbb{R}^n})$

$\Rightarrow L(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$ (add $-L(0_{\mathbb{R}^n})$ to both sides)