Symmetrizing polytopes and posets

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AMS Sectional Meeting

San Luis Obispo, CA
May 3–4, 2025

This is joint work with Federico Castillo.

PART I:

Motivation: Permutohedra, Associahedra and Permuto-Associahedra

Polytopes

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Remark: The face poset $\mathcal{F}(P)$ captures combinatorial properties of the polytope P without specifying its geometric properties.

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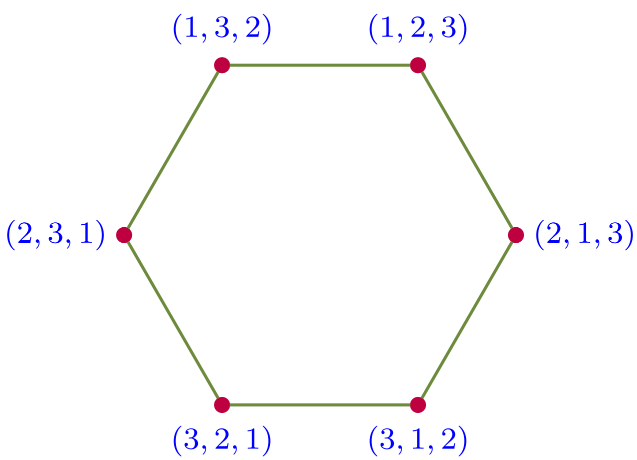
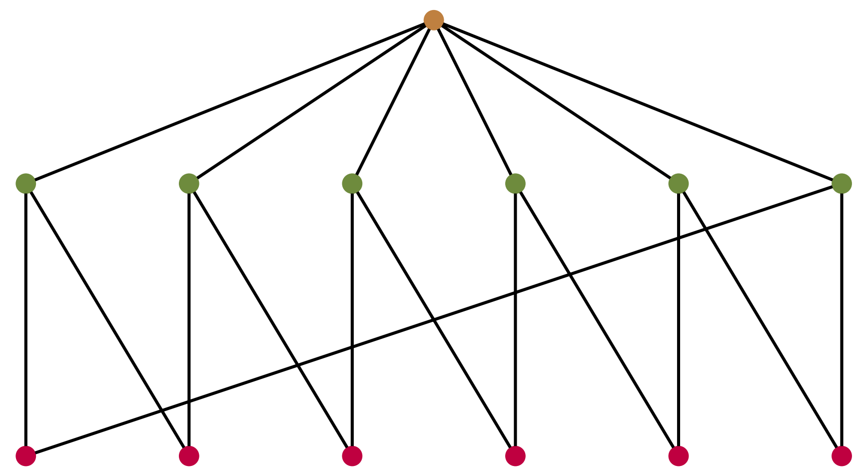


Π_2



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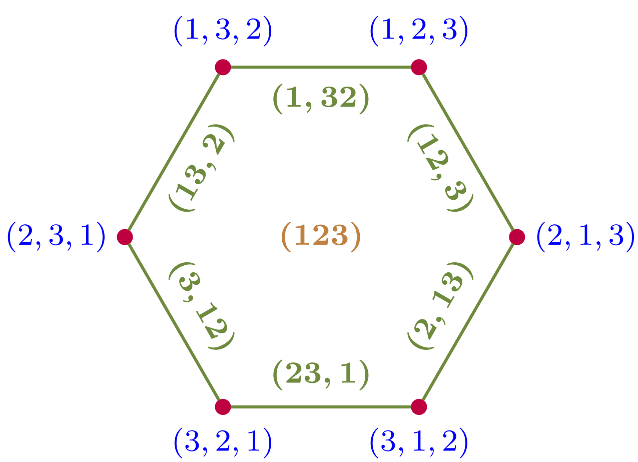

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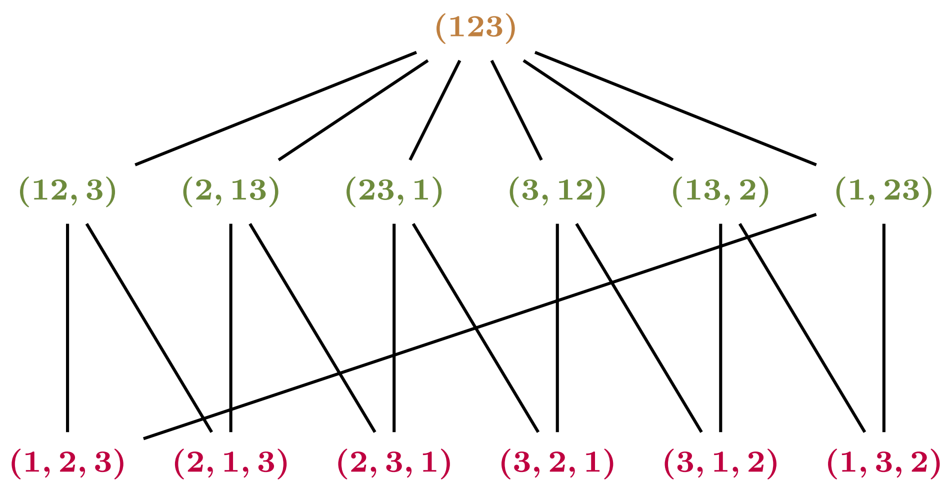

$\mathcal{F}(\Pi_2) \cong \mathscr{O}_3$

It is well-known that

$$\mathcal{F}(\Pi_d) \cong \mathscr{O}_{d+1},$$

the poset on *ordered (set) partitions* of $[d+1]$ ordered by "*merging blocks*".

Realization problem

Given a "nice" poset $\mathcal{F}$, the following is a classical question to ask:

Does there exist a polytope P such that $\mathcal{F} \cong \mathcal{F}(P)$?

If the answer is yes, we say $\mathcal{F}$ is *realizable*, and such a polytope P a *(geometric) realization* of $\mathcal{F}$.

Realizing associahedra

Definition. Let $\mathscr{K}_n$ be the poset on all "valid" bracketings on $(1 * 2 * \cdots * n)$ where the ordering is defined by "*removing brackets*".

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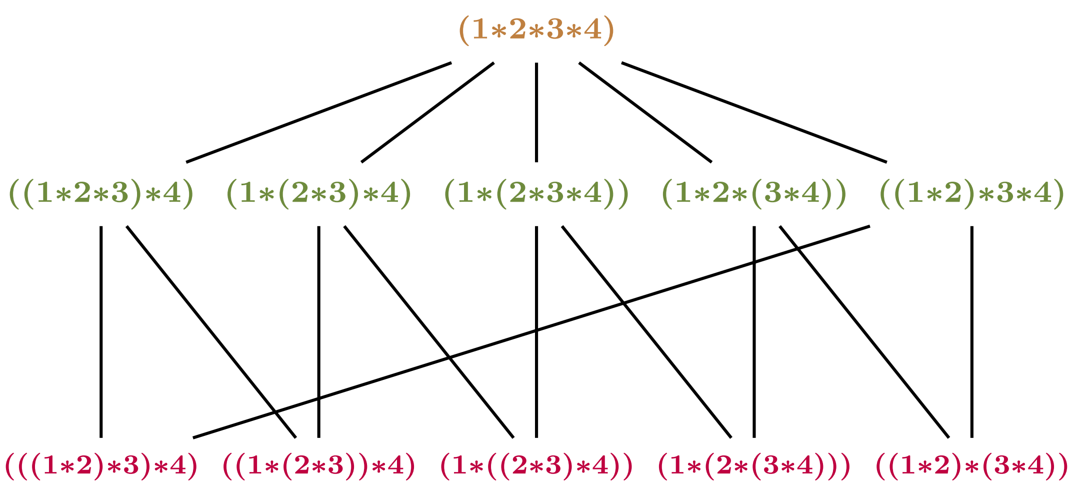
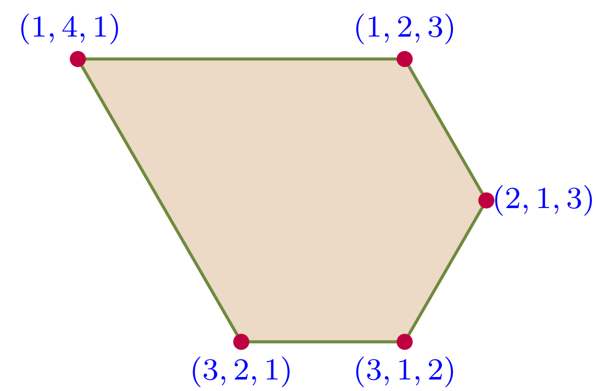


$\mathscr{K}_4$



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Kapranov's poset and permuto-associahedron

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(1) defined a poset $\mathcal{KA}_{d+1}$ which is a hybrid between $\mathscr{O}_{d+1}$ and $\mathscr{K}_{d+1}$, the face posets of the permutohedron and the associahedron.

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  Below is one maximal chain in $\mathcal{KA}_3$:

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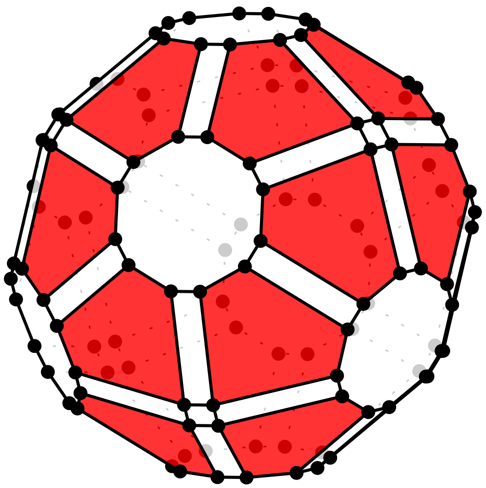
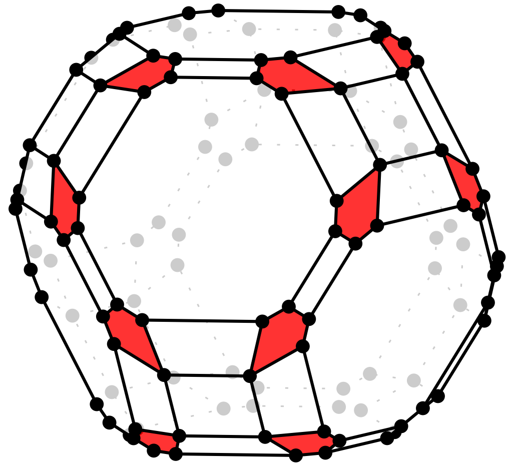
(3) asked whether $\mathcal{KA}_{d+1}$ can be realizable as a polytope. Such a polytope is called a *permuto-associahedron*.

Realizations of Permuto-associahedra



Reiner-Ziegler



Castillo-L.

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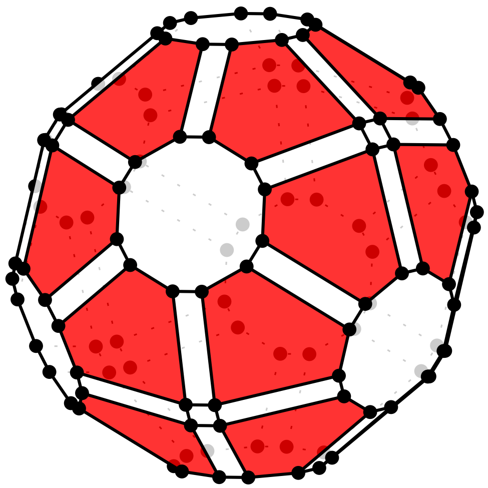

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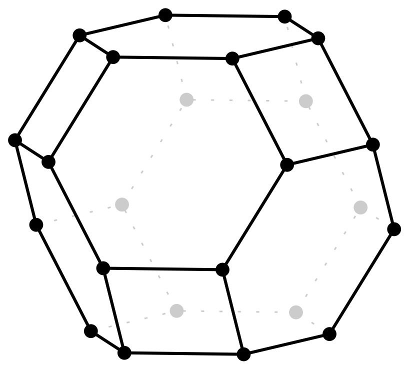
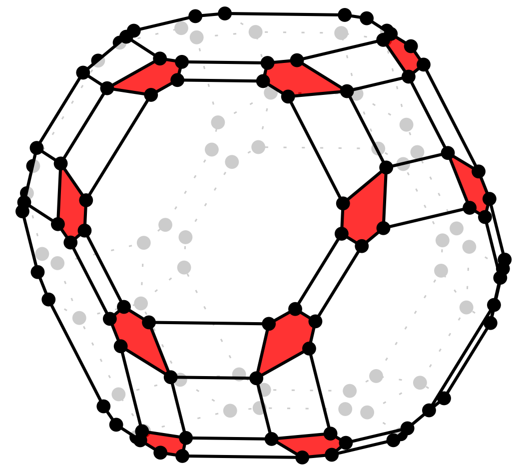


Π_3



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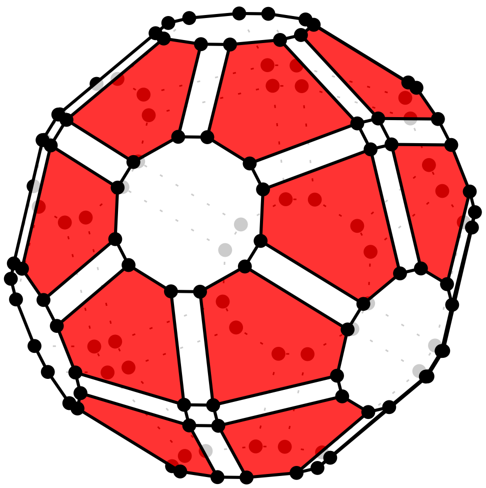

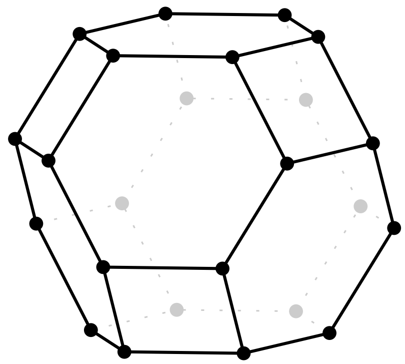

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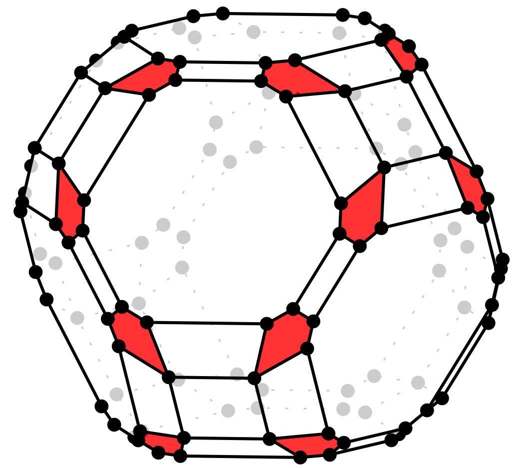

Both Reiner-Ziegler's and our construction of permuto-associahedron can be considerd as a $\mathfrak{S}_{d+1}$-symmetrization of a carefully embedded associahedron.

Questions

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- What if we use other reflection group $\mathfrak{G}$ to "symmetrize" polytopes?

PART II:

Symmetrization

Finite Reflection Groups

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  - It induces a simplicial fan on W that we denote $\Sigma(\mathcal{H})$.
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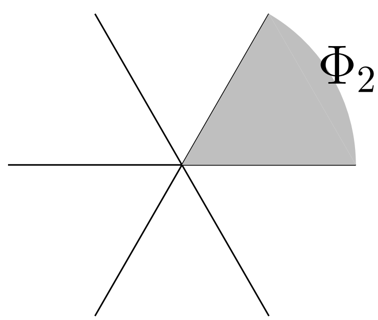
Example. The type-A finite reflection group is isomorphic to $\mathfrak{S}_{d+1}$. It arises from the *braid arrangement* $\mathcal{A}_d$ which induces the *braid fan* $\Sigma(\mathcal{A}_d)$.



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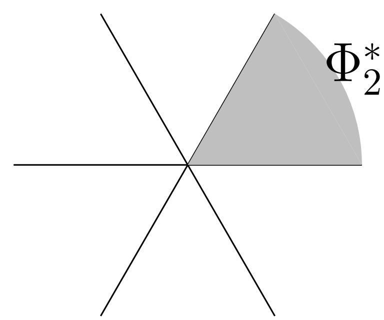

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Symmetrization

Definition. Let $\mathfrak{G}$ be a finite reflection group and P a polytope in U satisfying certain embedding conditions. The $\mathfrak{G}$*-symmetrization* of P is

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Our main goal is to find the connection between the combinatorics of $\mathfrak{G}(P)$ and that of P. This includes:

- Recover the combinatorics of P from that of $\mathfrak{G}(P)$.
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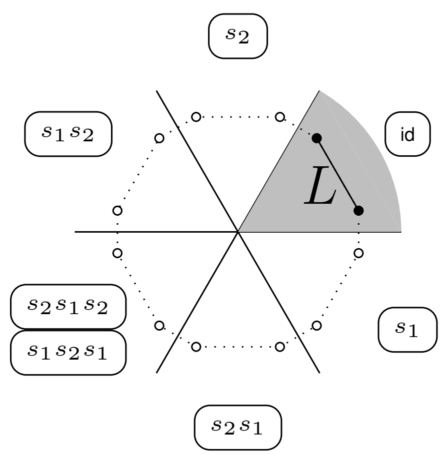
Note: Since for any polytope Q, its face poset $\mathcal{F}(Q)$ is dual to the face poset $\mathcal{F}(\Sigma(Q))$ of its normal fan $\Sigma(Q)$, we study the normal fans instead.

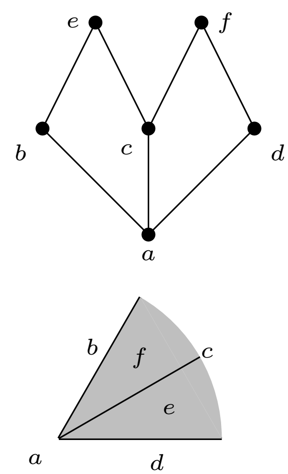
Fundamental Fan

Definition. The *fundamental fan* of P, and denote by $\mathrm{FFan}(P)$ is the (non-complete) fan induced by the intersections $\{\sigma \cap \Phi \ : \ \sigma \in \Sigma(P)\}$.

We denote by $\mathcal{Z}(P)$ the face poset of the fundamental fan $\mathrm{FFan}(P)$.

Example. Consider the line segment L shown on the left of the figure below.





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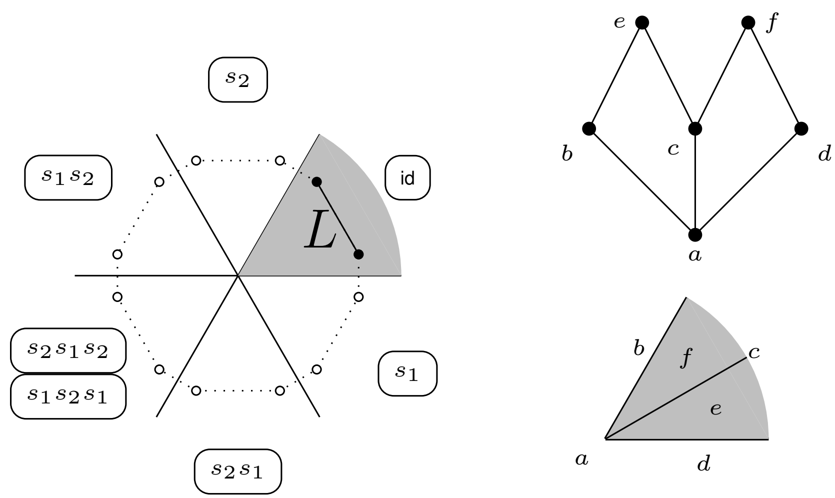

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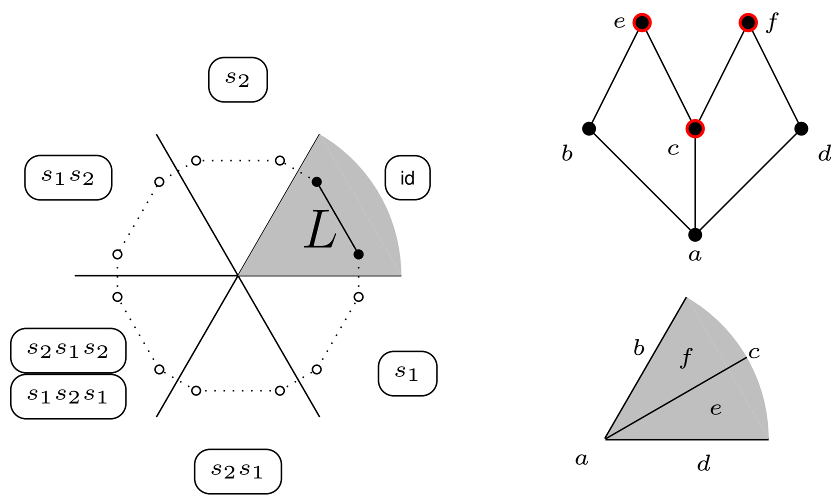

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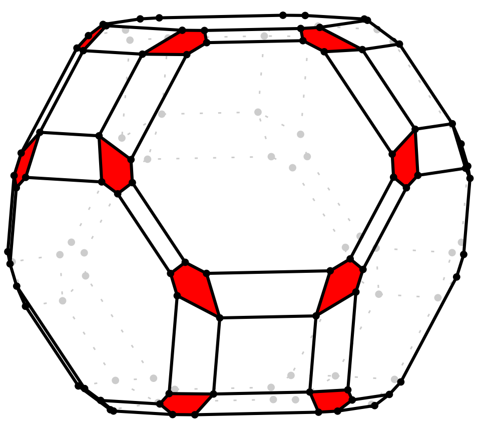
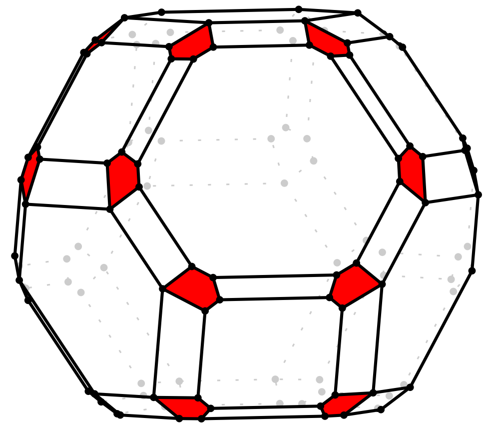
Fundamental Fan (cont'd)

Unfortunately, $\mathcal{Z}(P)$ does **not** determines the combinatorics of $\mathfrak{G}(P)$.

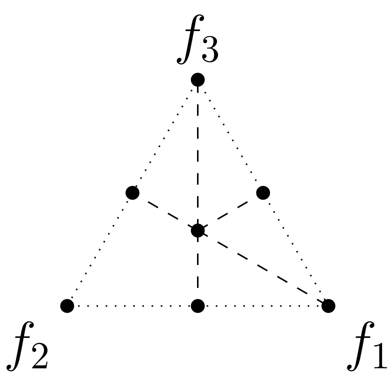
Example. The following two symmetrizations of a pentagon are combintorially different, while they have combinatorial equivalent fundamental fans.

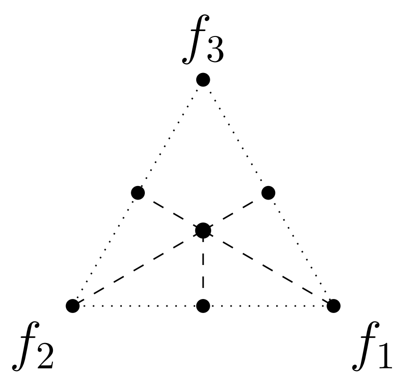


(a) Permuto–associahedron.



(b) Other.

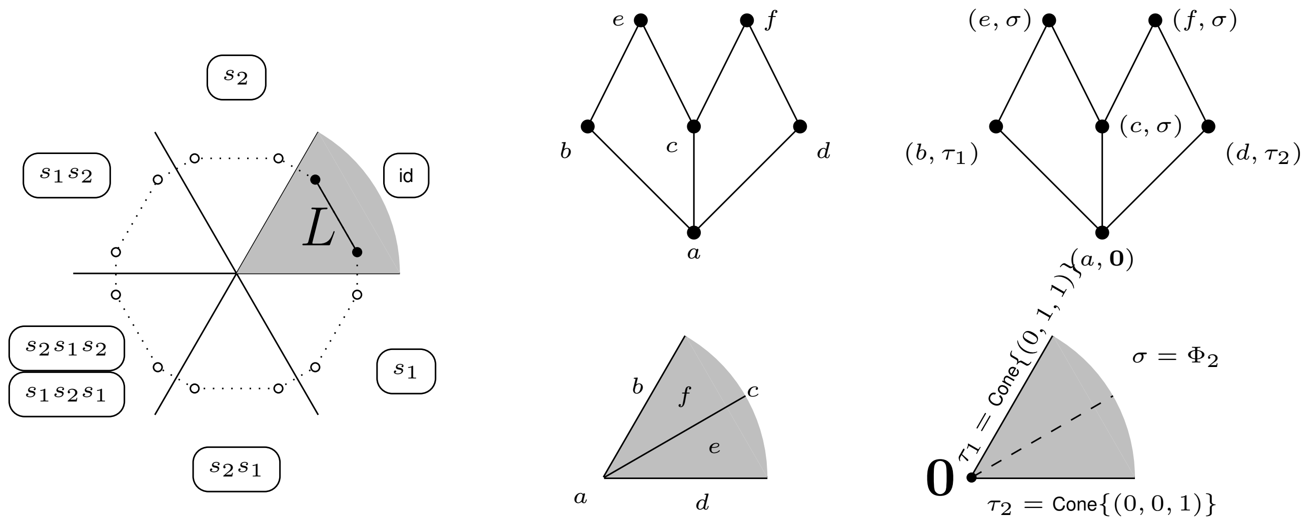




Refined Fundamental Fan

This motivates us to introduce the *refined fundamental fan* of P by separating cones in $\mathrm{FFan}(P)$ into different sets according to which face of Φ they “belong to”. We denote its face poset by $\mathcal{R}(P)$.

Example. Consider the line segment L again.



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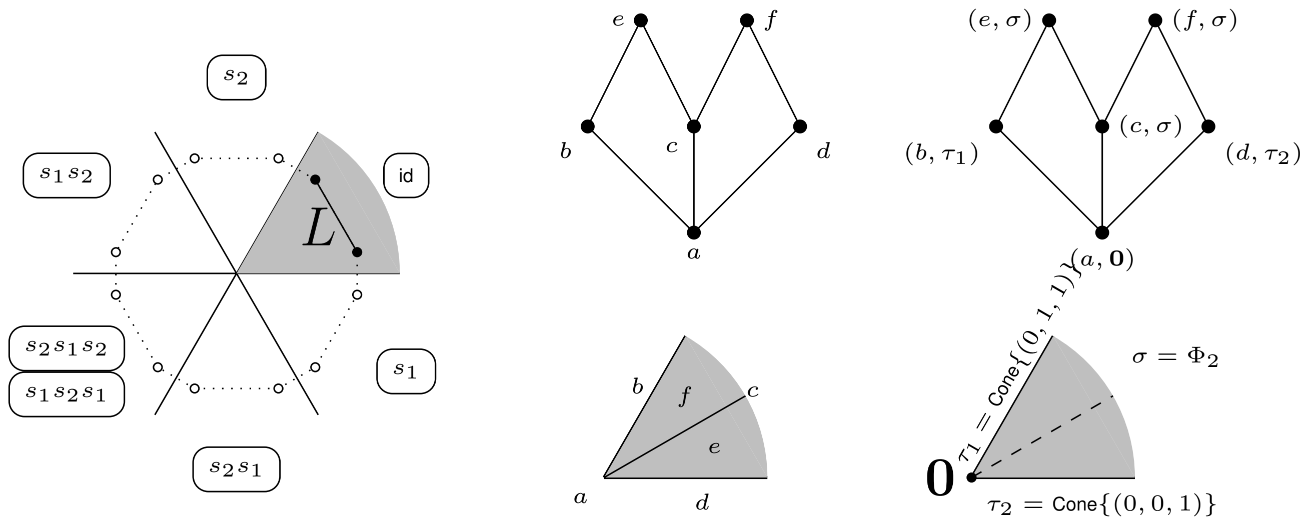

Theorem (Castillo-L.). *The $\mathfrak{G}$-symmetrization of $\mathcal{R}(P)$ is isomorphic to the dual of $\mathcal{F}(\mathfrak{G}(P))$, so it determines the combinatorics of $\mathfrak{G}(P)$.*

Application: Realizing a $\mathfrak{G}$-symmetric poset

Given a $\mathfrak{G}$-symmetric poset $\mathcal{F}$, e.g., Kapranov's poset $\mathcal{KA}_{d+1}$ is a $\mathfrak{S}_{d+1}$-symmetric poset.

Want to realize $\mathcal{F}$ as a $\mathfrak{G}$-symmetrization of some polytope P.

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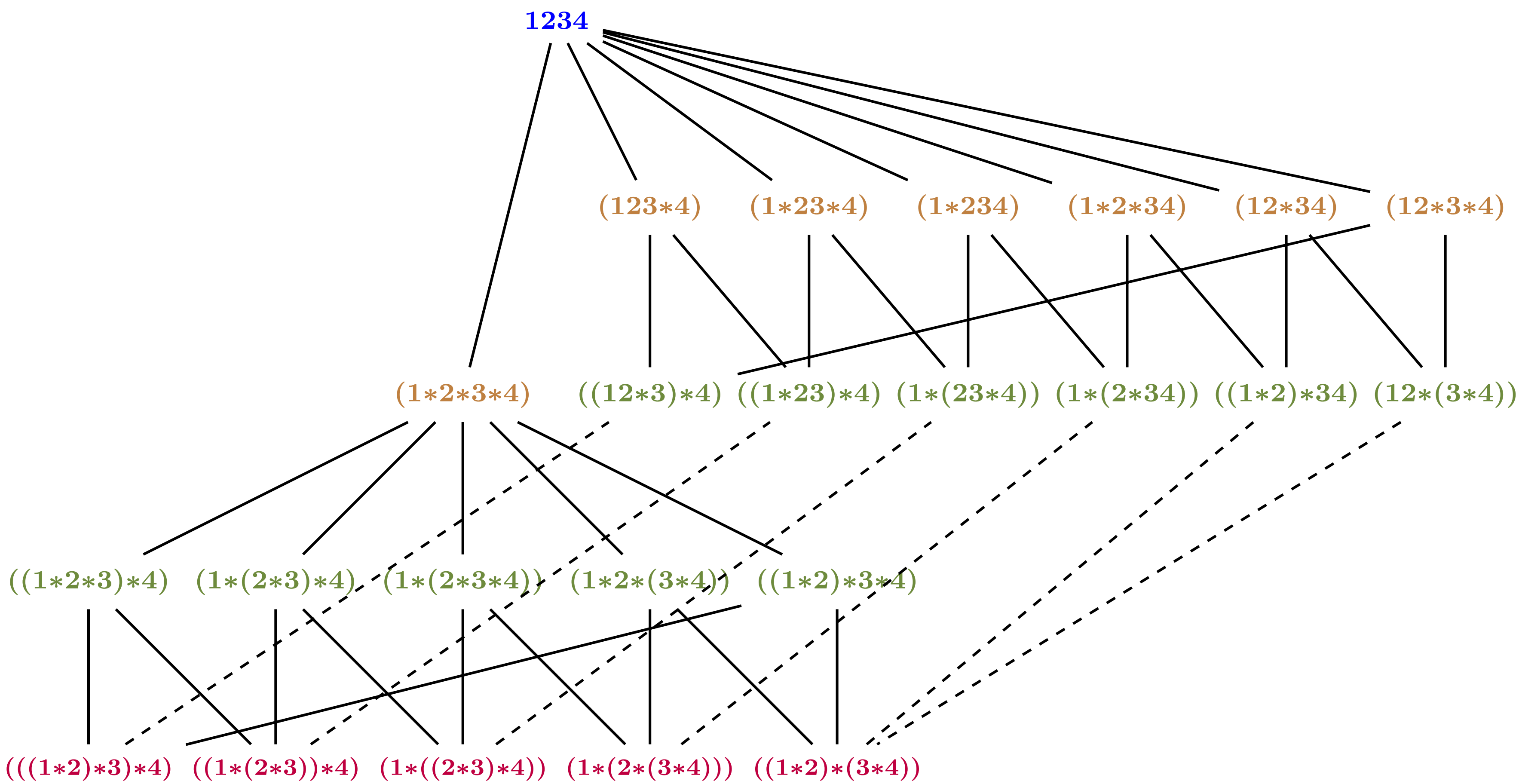
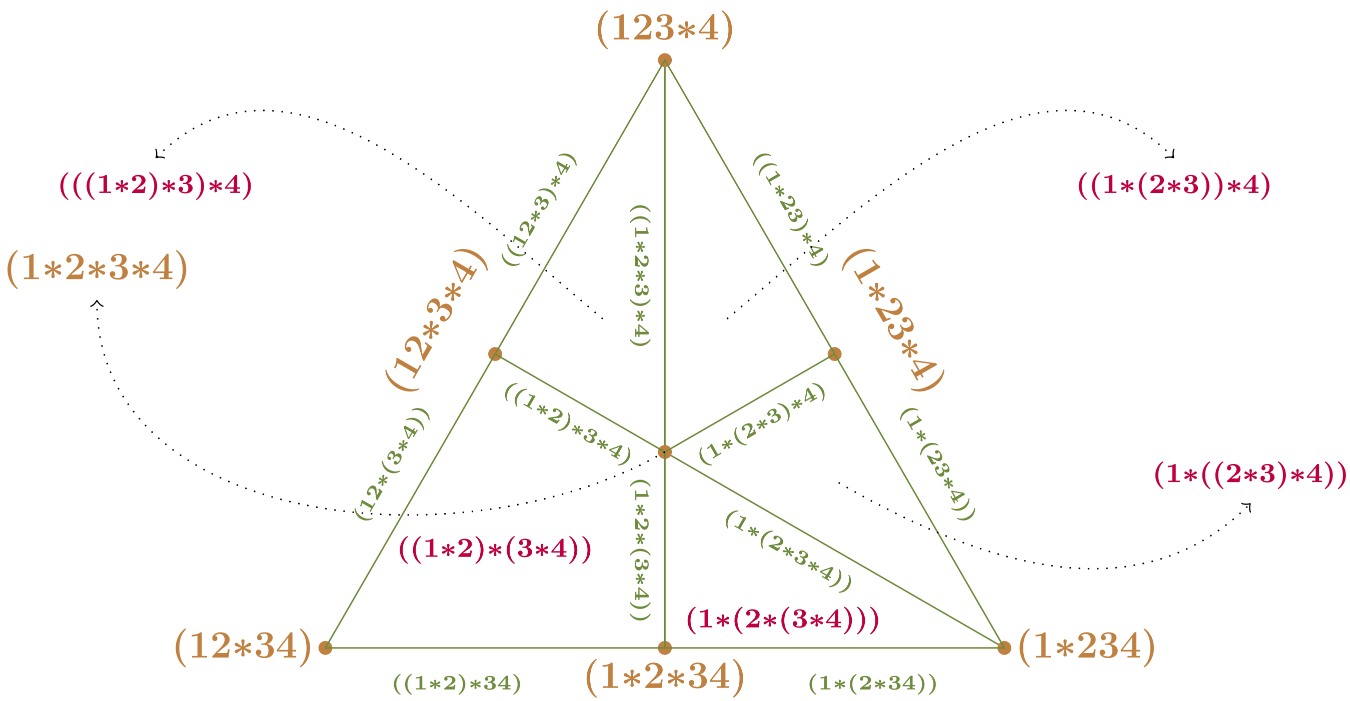
Main Idea:

- Find a $\mathfrak{G}$*-generator* $\mathcal{T}$ of $\mathcal{F}$, that is "compatible" with Φ.
- Our results reduce the original realization problem to realizing the dual of $\mathcal{T}$ as the fundamental fan of some polytope P.

Generator of Kapranov's poset $\mathcal{KA}_3$



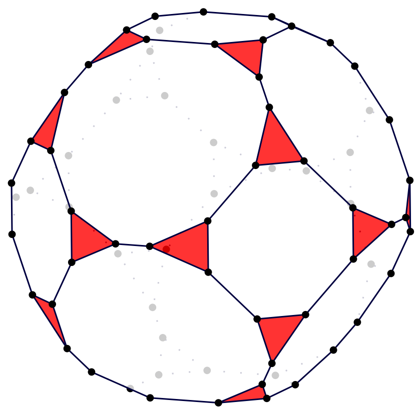
Realizing the generator of Kapranov's poset



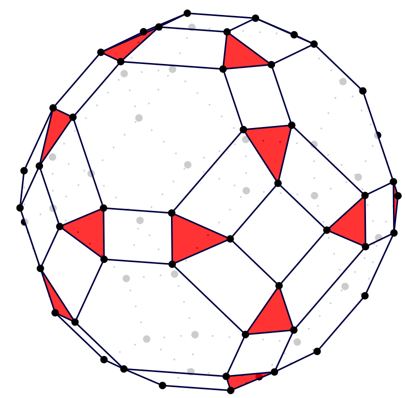
THANK YOU!

More Questions

Example. The $\mathfrak{S}_4$-symmetrizations of a triangle embeded in two different ways.



Simple



Not simple

Question. How many different combinatorial types are there?