

# Solutions

1 (12 pts.) Determine whether each statement is true (T) or false (F). Then CIRCLE the appropriate answer. You do not need to show work. Assume that  $x$  and  $y$  are positive numbers.

(a)  $\ln(x/y) = \ln x - \ln y$       T

(b)  $\frac{e^x}{e^y} = e^{x-y}$       T

(c)  $(\ln x)^2 = 2(\ln x)$       F

2 (10 pts.) Find the equation of the line tangent to  $y = \ln\left(\frac{1}{1+e^{x-1}}\right)$  at the point  $(1, -\ln 2)$ .

**Solution:**

$$y' = (1 + e^{x-1}) \frac{d}{dx} \left( \frac{1}{1 + e^{x-1}} \right) = (1 + e^{x-1}) \left( -\frac{1}{(1 + e^{x-1})^2} \right) \frac{d}{dx} (1 + e^{x-1}) = -\frac{e^{x-1}}{1 + e^{x-1}}.$$

Thus, the slope of the tangent line at  $(1, -\ln 2)$  is  $y'(1) = -\frac{e^0}{1+e^0} = -\frac{1}{2}$  and the equation of the tangent line is

$$y = -\frac{1}{2}(x - 1) - \ln 2 = -\frac{1}{2}x + \frac{1}{2} - \ln 2.$$

- 3 (15 pts.)** The number of a certain type of bacteria increases continuously at a rate proportional to the number present. There are 100 present at a given time and 800 present 3 hours later.
- (a) Give a function  $P(t)$  to model the number of bacteria  $t$  hours after the initial time.
  - (b) How many will there be 10 hours after the initial time?
  - (c) How long will it take for the population to triple?

**Solution:**

- (a)  $P(t)$  is an exponential growth model. Thus,  $P(t) = Ce^{kt}$ , for some constants  $C, k$ . By the conditions given, we have

$$100 = P(0) = Ce^0 \Rightarrow C = 100,$$

$$800 = P(3) = Ce^{3k} = 100e^{3k} \Rightarrow e^{3k} = 8 \Rightarrow 3k = \ln 8 = 3 \ln 2 \Rightarrow k = \ln 2.$$

Hence,

$$P(t) = 100e^{\ln 2t} = 100 \cdot 2^t.$$

(b)  $P(10) = 100 \cdot 2^{10} = 102400.$

(c)

$$P(t) = 100 \cdot 2^t = 100 \cdot 3 \Rightarrow 2^t = 3 \Rightarrow t = \log_2 3 = \frac{\ln 3}{\ln 2}.$$

- 4 (18 pts.) Sketch the graph of the function  $f(x) = \ln(x^2 + 1) - 2$ . Determine extrema, points of inflection, axes intercepts and asymptotes.

**Solution:**

$$f'(x) = \frac{2x}{x^2 + 1}.$$

$$f'(x) = 0 \Rightarrow x = 0.$$

$$f''(x) = 2 \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = 2 \frac{1 - x^2}{(x^2 + 1)^2}.$$

$$f''(0) = 2 \frac{1 - 0}{1} = 2 > 0 \Rightarrow x = 0 \text{ is a minimum.}$$

$$f''(x) = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow \text{The possible points of inflection are } x = \pm 1.$$

$$f''(x) \begin{cases} > 0, & \text{if } |x| < 1, \text{ i.e. } -1 < x < 1; \\ < 0, & \text{if } |x| > 1, \text{ i.e. } x < -1 \text{ or } x > 1. \end{cases}$$

Hence, the concavity changes at  $x = \pm 1$ . We conclude that  $x = \pm 1$  are the points of inflection.

$$y = f(0) = \ln(0^2 + 1) - 2 = -2 \text{ is the y-intercept.}$$

$$f(x) = 0 \Rightarrow \ln(x^2 + 1) = 2 \Rightarrow x^2 + 1 = e^2 \Rightarrow x = \pm \sqrt{e^2 - 1} \text{ are the x-intercepts.}$$

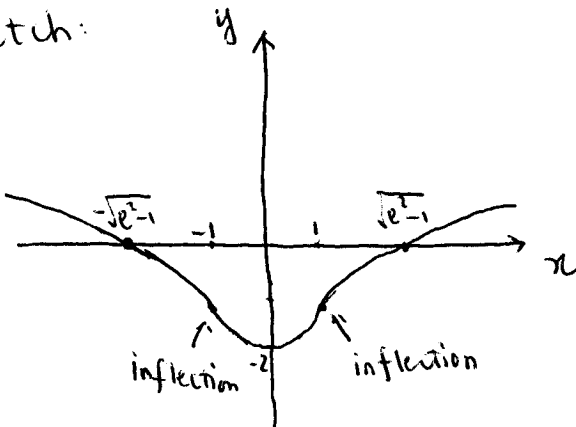
$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \ln(x^2 + 1) - 2 = \infty.$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \ln(x^2 + 1) - 2 = \infty.$$

Thus, there is no horizontal asymptote.

Because  $f(x)$  is defined and continuous on  $(-\infty, \infty)$ , for any real number  $x_0$ ,  $\lim_{x \rightarrow x_0} f(x) = f(x_0) \neq \infty$  or  $-\infty$ . Hence, there is no vertical asymptote.

Sketch:



- 5 (20 pts.) A ball is thrown upward with an initial velocity of 20 meters per second from an initial height of 10 meters. (The acceleration due to gravity is -10 meters per second squared.)
- (a) Derive formulas for the velocity (in meters/second) and height (in meters) of the ball as functions of the time  $t$  (in seconds.)
- (b) How long will it take for the ball to reach the maximum height? What is the maximum height?
- (c) How long will it take for the ball to hit the ground? What is the velocity of the ball when it hits the ground?

**Solution:**

- (a) Let  $h(t)$  and  $v(t)$  be the functions to describe the height and the velocity of the ball  $t$  seconds after the initial time. Then we have:

$$h'(t) = v(t), \quad h(0) = 10, \quad v'(t) = -10, \quad v(0) = 20.$$

$$v'(t) = -10 \Rightarrow v(t) = \int -10 \, dt = -10t + C_1.$$

$$20 = v(0) = -10 \cdot 0 + C_1 = C_1.$$

Hence, the velocity function is  $v(t) = -10t + 20$ .

$$h'(t) = v(t) = -10t + 20 \Rightarrow h(t) = \int (-10t + 20) \, dt = -5t^2 + 20t + C_2.$$

$$10 = h(0) = -5 \cdot 0^2 + 20 \cdot 0 + C_2 = C_2.$$

Thus, the height function is  $h(t) = -5t^2 + 20t + 10$ .

- (b)  $v(t) = 0 \Rightarrow -10t + 20 = 0 \Rightarrow t = 2$ . Hence, the ball reaches the maximum height after 2 seconds and the maximum height is  $h(2) = -5 \cdot 2^2 + 20 \cdot 2 + 10 = 30$  meters.
- (c)  $h(t) = 0 \Rightarrow -5t^2 + 20t + 10 = 0 \Rightarrow t^2 - 4t - 2 = 0 \Rightarrow t = 2 \pm \sqrt{6}$ . We only keep the positive solution. So it takes  $2 + \sqrt{6}$  seconds for the ball to reach the ground. The velocity of the ball when it hits the ground is  $v(2 + \sqrt{6}) = -10(2 + \sqrt{6}) + 20 = -10\sqrt{6}$  m/s.

6 (25 pts.) Evaluate the following indefinite integrals. Simplify your answers as much as possible. Show your work.

(a)

$$\begin{aligned} \int (3x^2 + 4x)\sqrt{x^3 + 2x^2 - 1} \, dx & \stackrel{u:=x^3+2x^2-1}{=} \int u^{\frac{1}{2}} \frac{du}{dx} \, dx \\ & = \frac{u^{1+\frac{1}{2}}}{1+\frac{1}{2}} + C \\ & = \frac{2}{3}(x^3 + 2x^2 - 1)^{\frac{3}{2}}. \end{aligned}$$

(b)

$$\begin{aligned} \int \frac{x^3 - x^2 + 2x - 1}{x + 2} \, dx & = \int (x^2 - 3x + 8) - \frac{17}{x + 2} \, dx \\ & = \int x^2 \, dx - 3 \int x \, dx + \int 8 \, dx - 17 \int \frac{1}{x + 2} \, dx \\ & = \frac{x^3}{3} - \frac{3x^2}{2} + 8x - 17 \ln|x + 2| + C. \end{aligned}$$

(c)

$$\begin{aligned} \int \frac{1 - e^{-x}}{1 + e^{-x}} \, dx & = \int \left( \frac{2}{1 + e^{-x}} - 1 \right) \, dx \\ & = 2 \int \frac{1}{1 + e^{-x}} \, dx - \int 1 \, dx = 2 \int \frac{e^x}{e^x + 1} \, dx - \int 1 \, dx \\ & \stackrel{u:=e^x+1}{=} 2 \int \frac{1}{u} \frac{du}{dx} \, dx - \int 1 \, dx = 2 \ln|u| - x + C \\ & = 2 \ln(e^x + 1) - x + C. \end{aligned}$$

(d)

$$\begin{aligned} \int \frac{1 + 3x^3}{x} e^{\ln x + x^3} \, dx & \stackrel{u:=\ln x + x^3}{=} \int e^u \frac{du}{dx} \, dx \\ & = e^u + C = e^{\ln x + x^3} + C. \end{aligned}$$

(e)

$$\begin{aligned} \int \frac{1}{x \ln x} \, dx & \stackrel{u:=\ln x}{=} \int \frac{1}{u} \frac{du}{dx} \, dx \\ & = \ln|u| + C = \ln|\ln x| + C. \end{aligned}$$