MAT 16B Prof. Fu Liu Practice Exam 1 April 2008

Solutions

1 (12 pts.) Determine whether each statement is true (T) or false (F). Then CIRCLE the appropriate answer. Assume that x and y are positive numbers.

(a) $\ln(x + y) = \ln x + \ln y$ F

(b) $e^x e^y = e^{xy}$ F

(c) $\int \ln x \, dx = \frac{1}{x} + C$ F

2 (10 pts.) Find the derivative of the function $f(x) = (x^2 + 2x - 1)^{\ln x}$.

<u>Solution</u>:

$$f(x) = (x^2 + 2x - 1)^{\ln x} = e^{(\ln x)\ln(x^2+2x-1)}$$

$$\begin{aligned}
f'(x) &= e^{(\ln x)\ln(x^2+2x-1)} \frac{d}{dx}((\ln x)\ln(x^2 + 2x - 1)) \\
&= (x^2 + 2x - 1)^{\ln x}(\ln x \frac{d}{dx}(\ln(x^2 + 2x - 1)) + \ln(x^2 + 2x - 1)\frac{d}{dx}(\ln x)) \\
&= (x^2 + 2x - 1)^{\ln x}(\frac{\ln x}{x^2 + 2x - 1}\frac{d}{dx}(x^2 + 2x - 1) + \frac{1}{x}\ln(x^2 + 2x - 1)) \\
&= (x^2 + 2x - 1)^{\ln x}(\frac{\ln x}{x^2 + 2x - 1}(2x + 2) + \frac{1}{x}\ln(x^2 + 2x - 1)) \\
&= (x^2 + 2x - 1)^{\ln x}(\frac{(2x + 2)\ln x}{x^2 + 2x - 1} + \frac{\ln(x^2 + 2x - 1)}{x})
\end{aligned}$$

3 (15 pts.) A company produces a product for which the marginal cost of producing x units is modeled by $\frac{dC}{dx} = 2x - 12$ and the fixed cost (the cost when $x = 0$) is \$125.

(a) Find the total cost function and the average cost function.

(b) Find the total cost of producing 50 units.

Solution:

(a) First do the indefinite integral: $C(x) = \int(2x - 12)\ dx = x^2 - 12x + K$. Then use the initial condition: $125 = C(0) = 0^2 - 12 \cdot 0 + K \Rightarrow K = 125$. Hence,the cost function is $C(x) = x^2 - 12x + 125$. And the average cost function is $A(x) = C(x)/x = x - 12 + \frac{125}{x}$.

(b) $C(50) = 50^2 - 12 \cdot 50 + 125 = 2500 - 600 + 125 = 2025$(dollars).

4 (20 pts.) You deposit \$500 in account AA paying 6% nominal interest rate compounded monthly, and deposit \$400 in account BB paying 7% nominal interest rate compounded continuously. (You may use a calculator for this problem.)

(a) What is the approximate effective rate of each account?

(b) What is the approximate balance in each account after 5 years?

(c) When will the balance in account BB be doubled?

Solution:

(a) AA: $r_{eff} = (1 + \frac{0.06}{12})^{12} - 1 \approx 0.0617$. BB: $r_{eff} = e^{0.07} - 1 \approx 0.0725$.

(b) Let $P_A(t)$ and $P_B(t)$ be the functions to describe the balance in accounts AA and BB after t years. Then $P_A(t) = 500(1+\frac{0.06}{12})^{12t} = 500 \cdot 1.005^{12t}$ and $P_B(t) = 400e^{0.07t}$. Thus, after 5 years, we have $500 \cdot 1.005^{60} \approx 674.43$ dollars in account AA and $400e^{0.07 \cdot 5} = 400 \cdot e^{0.35} \approx 567.63$ dollars in account BB.

(c) $P_B(t) = 400e^{0.07t} = 400 \cdot 2 \Rightarrow e^{0.07t} = 2 \Rightarrow t = \frac{\ln 2}{0.07}$.

5 (18 pts.) Analyze the function $f(x) = (x+3)^2 e^{-x}$ to determine extrema, points of inflection, axes intercepts and asymptotes. (Hint: $\lim_{y\to\infty} \frac{y^2}{e^y} = 0$.)

<u>**Solution**</u>:

$f'(x) = 2(x+3)e^{-x} - (x+3)^2 e^{-x} = (2-(x+3))(x+3)e^{-x} = -(x+1)(x+3)e^{-x} = -(x^2+4x+3)e^{-x}$.

$$f'(x) = 0 \Rightarrow x = -1 \text{ or } x = -3.$$

$$f''(x) = -(2x+4)e^{-x} + (x^2+4x+3)e^{-x} = (x^2+2x-1)e^{-x}.$$

$$f''(-1) = (1-2-1)e^1 < 0 \Rightarrow x = -1 \text{ is a maximum.}$$

$$f''(-3) = (9-6-1)e^3 > 0 \Rightarrow x = -3 \text{ is a minimum.}$$

$$f''(x) = 0 \Rightarrow x^2+2x-1 = 0 \Rightarrow \text{ The possible points of inflection are } x = -1 \pm \sqrt{2}.$$

After checking the concavity actually changes at $x = -1 \pm \sqrt{2}$, we conclude that $x = -1 \pm \sqrt{2}$ are the points of inflection.

$$y = f(0) = 3^2 e^0 = 9 \text{ is the y-intercept.}$$

$$f(x) = 0 \Rightarrow (x+3)^2 = 0 \Rightarrow x = -3 \text{ is the x-intercept.}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x+3)^2}{e^x} = 0.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} (-x+3)^2 e^x = +\infty.$$

Thus, $y = 0$ or the x-axis is a horizontal asymptote.

There does not exists x_0, such that $\lim_{x\to x_0} f(x) = \infty$. Hence, there is no vertical asymptote.

6 (25 pts.) Evaluate the following indefinite integrals. Simplify your answers as much as possible. Show your work.

(a)

$$\begin{aligned}\int \frac{x^3+10}{x+3}\,dx &= \int (x^2-3x+9-\frac{17}{x+3})\,dx\\ &= \int x^2\,dx - 3\int x\,dx + \int 9\,dx - 17\int \frac{1}{x+3}\,dx\\ &= \frac{x^3}{3} - \frac{3x^2}{2} + 9x - 17\ln|x+3| + C.\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{e^{-x}}{1-e^{-x}}\,dx &\overset{u:=1-e^{-x}}{=} \int \frac{1}{u}\frac{du}{dx}\,dx\\ &= \ln|u| + C\\ &= \ln|1-e^{-x}| + C.\end{aligned}$$

(c)

$$\begin{aligned}\int x(x^2-1)^5\,dx &\overset{u:=x^2-1}{=} \int \frac{1}{2}u^5\frac{du}{dx}\,dx\\ &= \frac{1}{2}\frac{u^6}{6} + C\\ &= \frac{1}{12}(x^2-1)^6 + C.\end{aligned}$$

(d)

$$\begin{aligned}\int x^2e^{x^3}\,dx &\overset{u:=x^3}{=} \int \frac{1}{3}e^u\frac{du}{dx}\,dx\\ &= \frac{1}{3}e^u + C\\ &= \frac{1}{3}e^{x^3} + C.\end{aligned}$$

(e)

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &\overset{u:=\ln x}{=} \int u\frac{du}{dx}\,dx\\ &= \frac{u^2}{2} + C\\ &= \frac{1}{2}(\ln x)^2 + C\end{aligned}$$