MAT 16B Prof. Fu Liu Practice Exam 2 May 2008

Solutions

1 (15 pts.) A company expects its income c during the next 4 years to be modeled by $c = 150,000 + 75,000t$.

(a) Find the expected actual income for the business over the 4 years.

(b) Assuming an annual inflation rate of 4%, what is the approximate present value of this income? (You may use a calculator for this part.)

<u>Solution</u>:

(a)

$$\begin{aligned}
\text{Actual income} &= \int_0^4 c(t)\ dt = \int_0^4 (150,000 + 75,000t)\ dt \\
&= [150,000t + 37,500t^2]_0^4 \\
&= (600,000 + 6,000,000) - (0 + 0) \\
&= 1,200,000(\text{dollars})
\end{aligned}$$

(b)

$$\begin{aligned}
\text{Present value} &= \int_0^4 c(t)e^{-rt}\ dt = \int_0^4 (150,000 + 75,000t)e^{-0.04t}\ dt \\
&= 150,000 \int_0^4 e^{-0.04t}\ dt + 75,000 \int_0^4 te^{-0.04t}\ dt \\
&= \frac{150,000}{-0.04}[e^{-0.04t}]_0^4 + \frac{75,000}{-0.04}\int_0^4 t\ d(e^{-0.04t}) \\
&= -3,750,000(e^{-0.16} - 1) - 1,875,000([te^{-0.04t}]_0^4 - \int_0^4 e^{-0.04t}\ dt) \\
&= -3,750,000e^{-0.16} + 3,750,000 - 1,875,000((4e^{-0.16} - 0) - \left[\frac{e^{-0.04t}}{-0.04}\right]_0^4) \\
&= -3,750,000e^{-0.16} + 3,750,000 - 1,875,000(4e^{-0.16} + 25(e^{-0.16} - 1)) \\
&= -58,125,000e^{-0.16} + 50,625,000 \\
&\approx 1,094,142(\text{dollars})
\end{aligned}$$

2 (10 pts.) Find the area of the region bounded by the graphs of $x = y^2 - 4$ and $x = 3y$.

<u>**Solution**</u>:The intersection points: $x = y^2 - 4$ and $x = 3y \Rightarrow (x, y) = (-3, -1), (12, 4)$.

$$(y^2 - 4) - 3y = (y + 1)(y - 4) < 0, \text{when } -1 < y < 4.$$

Thus, Area $= \int_{-1}^{4} (3y - (y^2 - 4))\, dy = [-\frac{y^3}{3} + \frac{3y^2}{2} + 4y]_{-1}^{4} = (-\frac{64}{3} + \frac{48}{2} + 16) - (\frac{1}{3} + \frac{3}{2} - 4) = 20\frac{5}{6}$.

3 (12 pts.) Let R be the region bounded by the graphs of $y = x^2, y = 0$ and $x = 2$. (For both parts, set up but do NOT evaluate the integral(s).)

(a) Find the volume of the solid formed by revolving R about the x-axis.

(b) Find the volume of the solid formed by revolving R about the y-axis.

<u>**Solution**</u>:

(a) The intersection of $y = x^2$ and $y = 0$ is the origin $(0, 0)$. Thus, $0 \leq x \leq 2$.

$$\text{Volume} = \pi \int_0^2 (x^2)^2\, dx = \pi \int_0^2 x^4\, dx.$$

(b) The intersection of $y = x^2$ and $x = 2$ is the point $(2, 4)$. Thus, $0 \leq y \leq 4$.
$y = x^2 \Rightarrow x = \sqrt{y}$.

$$\text{volume} = \pi \int_0^4 (2^2 - (\sqrt{y})^2)\, dy = \pi \int_0^4 (4 - y)\, dy.$$

4 (42 pts.) Evaluate the following indefinite integrals. Simplify your answers as much as possible. Show your work.

(a)

$$\begin{aligned}
\int x\ln(x+1)\;dx &\overset{u:=x+1}{=} \int (u-1)\ln u\;du = \int \ln u\; d(\frac{u^2}{2}-u) \\
&= (\frac{u^2}{2}-u)\ln u - \int (\frac{u^2}{2}-u)\;d(\ln u) \\
&= (\frac{u^2}{2}-u)\ln u - \int (\frac{u}{2}-1)\;du \\
&= (\frac{u^2}{2}-u)\ln u - \frac{u^2}{4}+u+C \\
&= (\frac{(x+1)^2}{2}-(x+1))\ln(x+1) - \frac{(x+1)^2}{4}+(x+1)+C \\
&= (x+1)(\frac{x-1}{2}\ln(x+1)-\frac{x-3}{4})+C
\end{aligned}$$

(b) $\int \frac{3x^2-7x-2}{x^3-x}\;dx$

$\frac{3x^2-7x-2}{x^3-x} = \frac{3x^2-7x-2}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1} = \frac{A(x^2-1)+B(x^2+x)+C(x^2-x)}{x^3-x}$.

$\Rightarrow 3x^2-7x-2 = A(x^2-1)+B(x^2+x)+C(x^2-x)$.

$x=0 \Rightarrow -2=-A \Rightarrow A=2$.

$x=1 \Rightarrow -6=2B \Rightarrow B=-3$.

$x=-1 \Rightarrow 8=2C \Rightarrow C=4$. Thus,

$$\int \frac{3x^2-7x-2}{x^3-x}\;dx = \int \frac{2}{x}-\frac{3}{x-1}+\frac{4}{x+1}\;dx = 2\ln|x|-3\ln|x-1|+4\ln|x+1|+C.$$

(c) $\int \frac{x}{\sqrt{2+3x}}\;dx$

Let $u=2+3x$, then $x=\frac{u-2}{3} \Rightarrow dx=\frac{1}{3}du$.

$$\begin{aligned}
\int \frac{x}{\sqrt{2+3x}}\;dx &\overset{u:=2+3x}{=} \int \frac{u-2}{3\sqrt{u}}\frac{1}{3}\;du \\
&= \frac{1}{9}\int (u^{1/2}-2u^{-1/2}\;du) \\
&= \frac{1}{9}(\frac{2}{3}u^{3/2}-2\cdot 2u^{1/2})+C \\
&= \frac{2}{27}(2+3x)^{3/2}-\frac{4}{9}(2+3x)^{1/2}+C
\end{aligned}$$

(d)

$$\begin{aligned}\int x\cot x^2\ dx &\overset{u:=x^2}{=} \frac{1}{2}\int \cot u\frac{du}{dx}\ dx\\ &= \frac{1}{2}\ln|\sin u| + C = \frac{1}{2}\ln|\sin(x^2)| + C.\end{aligned}$$

(e)

$$\begin{aligned}\int \frac{e^{1+\sqrt{x}}}{\sqrt{x}}\ dx &\overset{u:=1+\sqrt{x}}{=} 2\int e^u\frac{du}{dx}\ dx\\ &= 2e^u + C = 2e^{1+\sqrt{x}} + C.\end{aligned}$$

(f)

$$\begin{aligned}\int \frac{1}{x(\ln x)^5}\ dx &\overset{u:=\ln x}{=} \int u^{-5}\frac{du}{dx}\ dx\\ &= \frac{1}{-4}u^{-4} + C = -\frac{1}{4(\ln x)^4} + C.\end{aligned}$$

(g)

$$\begin{aligned}\int \sec^4 x\tan x\ dx &= \int \sec^3 x(\sec x\tan x)\ dx\\ &\overset{u:=\sec x}{=} \int u^3\frac{du}{dx}\ dx\\ &= \frac{1}{4}u^4 + C\\ &= \frac{1}{4}\sec^4 x + C.\end{aligned}$$

Or, alternatively

$$\begin{aligned}\int \sec^4 x\tan x\ dx &= \int \sec^2 x(1+\tan^2 x)\tan x\ dx\\ &\overset{u:=\tan x}{=} \int (1+u^2)u\frac{du}{dx}\ dx = \int (u^3+u)\ du\\ &= \frac{1}{4}u^4 + \frac{1}{2}u^2 + C\\ &= \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + C.\end{aligned}$$

5 (21 pts.) Evaluate the following definite integrals or improper integrals. (If it is an improper integral, determine whether it converges or not. If it does, then evaluate it.) Simplify your answers as much as possible. Show your work.

(a)

$$\begin{aligned}
\int_0^\pi x^2 \cos x \, dx &= \int_0^\pi x^2 \, d(\sin x) = [x^2 \sin x]_0^\pi - 2\int_0^\pi x \sin x \, dx) \\
&= (0-0) + 2\int_0^\pi x \, d(\cos x) \\
&= 2([x \cos x]_0^\pi - \int_0^\pi \cos x \, dx) \\
&= 2((-\pi - 0) - [\sin x]_0^\pi) = -2\pi.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^\infty x^2 e^{-x^3} \, dx &= \lim_{b\to\infty} \int_0^b x^2 e^{-x^3} \, dx \\
&= \lim_{b\to\infty} \frac{1}{-3} \int_0^b \frac{d(-x^3)}{dx} e^{-x^3} \, dx \\
&= \lim_{b\to\infty} -\frac{1}{3}[e^{-x^3}]_0^b = \lim_{b\to\infty} -\frac{1}{3}(e^{-b^3} - 1) \\
&= \frac{1}{3}
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^3 \frac{1}{(x-1)^2} \, dx &= \int_0^1 \frac{1}{(x-1)^2} \, dx + \int_1^3 \frac{1}{(x-1)^2} \, dx \\
&= \lim_{c\to 1^-} \int_0^c \frac{1}{(x-1)^2} \, dx + \lim_{d\to 1^+} \int_d^3 \frac{1}{(x-1)^2} \, dx \\
&= \lim_{c\to 1^-} \left[-\frac{1}{x-1}\right]_0^c + \lim_{d\to 1^+} \left[-\frac{1}{x-1}\right]_d^3 \\
&= \lim_{c\to 1^-} -\left(\frac{1}{c-1} - \frac{1}{-1}\right) + \lim_{d\to 1^+} -\left(\frac{1}{2} - \frac{1}{d-1}\right).
\end{aligned}$$

Both limits do not exist, so this improper integral diverges.