MAT 16B Prof. Fu Liu Practice Exam 3 June 2008

Solutions

1 (14 pts.) Fill in the following table for special probability density functions. Do not need to show the work.

Probability density function	Uniform	Exponential	Normal
$f(x)$	$\frac{1}{b-a}$	$ae^{-ax}, (a > 0)$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$
Range	$[a, b]$	$[0, \infty)$	$(-\infty, \infty)$
Expected value $E(x)$	$\frac{a+b}{2}$	$\frac{1}{a}$	μ
Variance $V(x)$	$\frac{(b-a)^2}{12}$	$\frac{1}{a^2}$	σ^2
Standard Deviation	$\frac{b-a}{2\sqrt{3}}$	$\frac{1}{a}$	σ
Median	$\frac{a+b}{2}$	$\frac{\ln 2}{a}$	μ

2 (6 pts.) Identify the type of following probability density functions, then use the table in problem 1 to find the mean, variance, standard deviation, and median without integration.

(a) $f(x) = \frac{1}{2}e^{-x/2}, x \in [0, \infty)$.

This is an exponential probability density function with $a = \frac{1}{2}$.

$$E(x) = 2,\ V(x) = 4,\ \sigma = 2,\ \text{median} = 2\ln 2.$$

(b) $f(x) = \frac{7}{3}, x \in [1, \frac{10}{7}]$.

This is a uniform probability density function with $a = 1$ and $b = \frac{10}{7}$.

$$E(x) = \frac{17}{14},\ V(x) = \frac{3}{196},\ \sigma = \frac{\sqrt{3}}{14},\ \text{median} = \frac{17}{14}.$$

(c) $f(x) = \frac{1}{\sqrt{98\pi}}e^{-(x-3)^2/98}, x \in (-\infty, \infty)$.

This is a normal probability density function with $\mu = 3$ and $\sigma = 7$.

$$E(x) = 3,\ V(x) = 49,\ \sigma = 7,\ \text{median} = 3.$$

3 (15 pts.) Given $f(x) = 2x - 2$ over $[1, 2]$:

(a) Verify $f(x)$ is a probability density function.

(b) Set up the integrals or formulas to find the mean, variance and standard deviation of the random variable. (Do NOT evaluate.)

(c) Find the median of the random variable.

(d) Set up the integral to find the probability of $1 < x < \frac{4}{3}$. (Do NOT evaluate.)

Solution:

(a) First, $f(x) = 2x - 2$ is continuous and nonnegative on the interval $[1, 2]$.
$\int_1^2 (2x - 2)\ dx = [x^2 - 2x]_1^2 = (4 - 4) - (1 - 2) = 1$.

(b) Mean: $\mu = E(x) = \int_1^2 xf(x)\ dx = \int_1^2 x(2x - 2)\ dx$.
Variance: $V(x) = E(x^2) - (E(x))^2 = \int_1^2 x^2(2x - 2)\ dx - \mu^2$.
Standard deviation: $\sigma = \sqrt{V(x)}$.

(c) $\frac{1}{2} = \int_1^m (2x - 2)\ dx = [x^2 - 2x]_1^m = (m^2 - 2m) - (1 - 2) = m^2 - 2m + 1$
$\Rightarrow 2m^2 - 4m + 1 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16 - 8}}{4} = \frac{2 \pm \sqrt{2}}{2}$.
Since the median has to be between 1 and 2, median $= \frac{2 + \sqrt{2}}{2}$.

(d) $P(1 < x < \frac{4}{3}) = \int_1^{\frac{4}{3}} (2x - 2)\ dx$.

4 (10 pts.) How large should n be in order to guarantee that the absolute error in the Trapezoidal Rule estimate of the following definite integral is at most 0.001?

$$\int_2^5 e^{x^2}\,dx \qquad\qquad \text{Recall: } |E| \leq \frac{(b-a)^3}{12n^2}\left[\max_{a\leq x\leq b}|f''(x)|\right].$$

(You may use a calculator for this problem.)

<u>**Solution**</u>:

$f(x) := e^{x^2},\ f'(x) = 2xe^{x^2},\ f''(x) = (4x^2+2)e^{x^2}$.

On $[2,5]$, $|f''(x)| = (4x^2+2)e^{x^2}$ is increasing. Hence,

$$\max_{2\leq x\leq 5}|f''(x)| = (4\cdot 5^2+2)e^{5^2} = 102e^{25}.$$

To make $|E| < 0.001$, it is sufficient to find n such that

$$\begin{aligned}
&\frac{(b-a)^3}{12n^2}\left[\max_{a\leq x\leq b}|f''(x)|\right] < 0.001\\
\Rightarrow\quad &\frac{(5-2)^3}{12n^2}102e^{25} < 0.001\\
\Rightarrow\quad &n^2 > \frac{(5-2)^3}{12\cdot 0.001}102e^{25} = 4500\cdot 51e^{25}\\
\Rightarrow\quad &n > \sqrt{4500\cdot 51e^{25}} \approx 128550085.17.
\end{aligned}$$

Therefore, $n \geq 128550086$.

5 (15 pts.) Set up the formulas but do NOT evaluate to estimate the definite integral

$$\int_1^4 \frac{1}{x^3+1}\,dx$$

by using midpoint rule, Trapezoidal rule, and Simpson's rule with $n = 6$.

Solution:

First, divide the interval [1, 4] into 6 small intervals of the same length $\Delta x = \frac{b-a}{n} = \frac{1}{2}$:

$$[1, \frac{3}{2}],\ [\frac{3}{2}, 2],\ [2, \frac{5}{2}],\ [\frac{5}{2}, 3],\ [3, \frac{7}{2}],\ [\frac{7}{2}, 4].$$

The midpoints of each interval are:

$$\frac{5}{4},\ \frac{7}{4},\ \frac{9}{4},\ \frac{11}{4},\ \frac{13}{4},\ \frac{15}{4}.$$

Midpoint rule:

$$\begin{aligned}
\int_1^4 \frac{1}{x^3+1}\,dx &\approx \Delta x \sum_i f(m_i) \\
&= \frac{1}{2}(f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4}) + f(\frac{13}{4}) + f(\frac{15}{4})) \\
&= \frac{1}{2}\left(\frac{1}{(\frac{5}{4})^3+1} + \frac{1}{(\frac{7}{4})^3+1} + \frac{1}{(\frac{9}{4})^3+1} + \frac{1}{(\frac{11}{4})^3+1} + \frac{1}{(\frac{13}{4})^3+1} + \frac{1}{(\frac{15}{4})^3+1}\right)
\end{aligned}$$

Trapezoidal rule:

$$\begin{aligned}
\int_1^4 \frac{1}{x^3+1}\,dx &\approx \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + f(x_6)) \\
&= \frac{1}{4}(f(1) + 2f(\frac{3}{2}) + 2f(2) + 2f(\frac{5}{2}) + 2f(3) + 2f(\frac{7}{2}) + f(4)) \\
&= \frac{1}{4}\left(\frac{1}{1^3+1} + 2\frac{1}{\frac{3}{2}^3+1} + 2\frac{1}{2^3+1}) + 2\frac{1}{\frac{5}{2}^3+1} + 2\frac{1}{3^3+1} + 2\frac{1}{\frac{7}{2}^3+1} + \frac{1}{4^3+1}\right)
\end{aligned}$$

Simpson's rule:

$$\begin{aligned}
\int_1^4 \frac{1}{x^3+1}\,dx &\approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)) \\
&= \frac{1}{6}(f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + f(4)) \\
&= \frac{1}{6}\left(\frac{1}{1^3+1} + 4\frac{1}{\frac{3}{2}^3+1} + 2\frac{1}{2^3+1} + 4\frac{1}{\frac{5}{2}^3+1} + 2\frac{1}{3^3+1} + 4\frac{1}{\frac{7}{2}^3+1} + \frac{1}{4^3+1}\right)
\end{aligned}$$