Math 108, ~~HW 1 solutions~~, cont'd

HW 2 Solutions

1.4

1.4 4(b) By (iv), the weapon was candlestick. By (iii), Scarlett is guilty.

5(f) **Proof.** There exist integers k, ℓ, so that $x = 2k+1$, $y = 2\ell+1$. Then

$$3x-5y = 3(2k+1) - 5(2\ell+1) = 6k - 10\ell - 2$$
$$= 2(3k - 5\ell - 1),$$

an even number. □

6(f). **Proof.** Case 1: $a \geq 0$. Then $|a| \leq b$ iff $a \leq b$, which holds by assumption.

Case 2: $a < 0$. Then $|a| \leq b$ iff $-a \leq b$, which also holds by assumption. □

8a. **Proof** Case 1: n even. Then $n = 2k$ for some $k \in \mathbb{Z}$, and $n^2+n+3 = 4k^2+2k+3 = 2(2k^2+k+1)+1$, an odd number.

Case 2: n odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$, and $n^2+n+3 = 4k^2+4k+1+2k+1+3$
$= 4k^2+6k+5 = 2(2k^2+3k+2)+1$, an odd number. □

9d. **Proof.** Note $2x+5 < 11$ iff $x < (11-5)/2 = 3$.

Further $x^3+2x^2 = x^2(x+2) < 0$ iff $x < -2$. But $x < -2$ implies $x < 3$. □

11(b). C, The proof needs this correction: ..., and for some integer r, $c = ar$. Then $b+c = a(q+r)$, so $a \mid (b+c)$. □

Math 108, Fall 2026

HW 2 solutions, cont'd

1.5 3g. **Proof** We prove that x odd implies $8 \mid (x^2-1)$. If x is odd, there is a $k \in \mathbb{Z}$ so that $x = 2k+1$. Then $x^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k+1)$. Either k or $k+1$ is even, so $4k(k+1)$ is divisible by 8. □

4c **Proof** Assume $x \leq 0$. Then $x^3 + x = x(x^2+1) \leq 0$, as x^2+1 is positive. □

5c **Proof** The distance $d((2,4),(0,3)) = \sqrt{4+1} = \sqrt{5}$, while $d((2,4),(3,1)) = \sqrt{1+9} = \sqrt{10} > \sqrt{5}$. □

6e **Proof** If $a \geq 2$, $a < b$ and $ab < 3$, then $b \geq 3$ (as $b > a$), and then $ab \geq 6$. □

7b **Proof** ($\Leftarrow$) Easy check.

($\Rightarrow$) If $b \mid (b+3)$, then $b+3 = \ell b$ for some $\ell \geq 1$, but then $3 = (\ell - 1)b$, and $b \mid 3$; as 3 is a prime so $b = 1$ or $b = 3$. But $b = 1$ is impossible, as $(a+1) \mid 1$ is impossible because $a+1 \geq 2$. Therefore $b = 3$; $a+1 \mid 3$ and $a+1 \geq 2$ then together imply $a+1 = 3$, $a = 2$. □

10. Omitted.

12(d) The proof is correct, but unnecessarily convoluted. Erase the first two sentences, replace next "However" by "By the assumption" and shorten the last sentence to "Thus a divides $b+c$".

HW2 Solutions, Cont'd

1.6

4(b) True. Take $y = -x$.

4(c) False. Counterexample: $x = 2$, $y = 1$.

4(g) False. Counterexample: $x = 1$.

5(a) Proof. If $x = 1$, both sides are false, so we will assume $x > 1$ from now on.

($\Rightarrow$) By contradiction. Assume x is a prime, but there exist an n, $1 < n \leq \sqrt{x}$, so that $n \mid x$. As x is a prime, $n \mid x$ and $n > 1$, $n = x$. But then $x \leq \sqrt{x}$, which is impossible when $x > 1$.

($\Leftarrow$) Contrapositive. Assume x is not a prime. Then $x = ab$, for some $a, b \in \mathbb{N}$, $a > 1$, $b > 1$. At least one of a, b must be less or equal $\sqrt{x}$: if both $a > \sqrt{x}$ and $b > \sqrt{x}$, then $a \cdot b > \sqrt{x} \cdot \sqrt{x} = x$, a contradiction with $a \cdot b = x$. Take n to be the minimum of a and b. Then $n \leq \sqrt{x}$, $n > 1$, and $n \mid x$. $\square$

6(i) Proof. The inequality $\frac{1}{r^2} < \frac{1}{100}$ is equivalent to $r > 10$. So we can take $K = 10$.

7(h) F. What is proved is the converse $(x = 0) \vee (y = 0) \Rightarrow xy = 0$.