Exercises 1.4

1. Analyze the logical form of each of the following statements and construct just the outline of a proof. Since the statements may contain terms with which you are not familiar, you should not (and perhaps could not) provide any details of the proof.

★ **(a)** Outline a direct proof that if $(G, *)$ is a cyclic group, then $(G, *)$ is abelian.

(b) Outline a direct proof that if **B** is a nonsingular matrix, then the determinant of **B** is not zero.

(c) Suppose A, B, and C are sets. Outline a direct proof that if A is a subset of B and B is a subset of C, then A is a subset of C.

(d) Outline a direct proof that if the maximum value of the differentiable function f on the closed interval $[a, b]$ occurs at x_0, then either $x_0 = a$ or $x_0 = b$ or $f'(x_0) = 0$.

(e) Outline a direct proof that if **A** is a diagonal matrix, then **A** is invertible whenever all its diagonal entries are nonzero.

2. A theorem of linear algebra states that if **A** and **B** are invertible matrices, then the product **AB** is invertible. As in Exercise 1, outline

(a) a direct proof of the theorem.

(b) a direct proof of the converse of the theorem.

3. Verify that $[(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)] \Rightarrow B$ is a tautology. See the example on page 30.

4. These facts have been established at a crime scene.

(i) If Professor Plum is not guilty, then the crime took place in the kitchen.

(ii) If the crime took place at midnight, Professor Plum is guilty.

(iii) Miss Scarlet is innocent if and only if the weapon was not the candlestick.

(iv) Either the weapon was the candlestick or the crime took place in the library.

(v) Either Miss Scarlet or Professor Plum is guilty.

Use the above and the additional fact(s) below to solve the case. Explain your answer.

★ **(a)** The crime lab determines that the crime took place in the library.

(b) The crime lab determines that the crime did not take place in the library.

(c) The crime lab determines that the crime was committed at noon with the revolver.

(d) The crime took place at midnight in the conservatory. (Give a complete answer.)

5. Let x and y be integers. Prove that

(a) if x and y are even, then $x + y$ is even.

(b) if x is even, then xy is even.

(c) if x and y are even, then xy is divisible by 4.

(d) if x and y are even, then $3x - 5y$ is even.

(e) if x and y are odd, then $x + y$ is even.

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(f) if x and y are odd, then $3x - 5y$ is even.
(g) if x and y are odd, then xy is odd.
★ **(h)** if x is even and y is odd, then $x + y$ is odd.
(i) if x is even and y is odd, then xy is even.

6. Let a and b be real numbers. Prove that
(a) $|ab| = |a||b|$.
(b) $|a - b| = |b - a|$.
(c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, for $b \neq 0$.
☆ **(d)** $|a + b| \leq |a| + |b|$.
(e) if $|a| \leq b$, then $-b \leq a \leq b$.
(f) if $-b \leq a \leq b$, then $|a| \leq b$.

7. Suppose a, b, c, and d are integers. Prove that
(a) $2a - 1$ is odd.
★ **(b)** if a is even, then $a + 1$ is odd.
(c) if a is odd, then $a + 2$ is odd.
☆ **(d)** $a(a + 1)$ is even.
(e) 1 divides a.
(f) a divides a.
★ **(g)** if a and b are positive and a divides b, then $a \leq b$.
(h) if a divides b, then a divides bc.
★ **(i)** if a and b are positive and $ab = 1$, then $a = b = 1$.
(j) if a and b are positive, a divides b and b divides a, then $a = b$.
(k) if a divides b and c divides d, then ac divides bd.
(l) if ab divides c, then a divides c.
(m) if ac divides bc, then a divides b.

8. Give two proofs that if n is a natural number, then $n^2 + n + 3$ is odd.
(a) Use two cases.
(b) Use Exercises 7(d) and 5(h).

9. Let a, b, and c be integers and x, y, and z be real numbers. Use the technique of working backward from the desired conclusion to prove that
(a) if x and y are nonnegative, then $\frac{x + y}{2} \geq \sqrt{xy}$.
Where in the proof do we use the fact that x and y are nonnegative?
(b) if a divides b and a divides $b + c$, then a divides $3c$.
(c) if $ab > 0$ and $bc < 0$, then $ax^2 + bx + c = 0$ has two real solutions.
(d) if $x^3 + 2x^2 < 0$, then $2x + 5 < 11$.
(e) if an isosceles triangle has sides of length x, y, and z, where $x = y$ and $z = \sqrt{2xy}$, then it is a right triangle.

10. Recall that except for degenerate cases, the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

an ellipse iff $B^2 - 4AC < 0$,
a parabola iff $B^2 - 4AC = 0$,
a hyperbola iff $B^2 - 4AC > 0$.

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★ **(a)** Prove that the graph of the equation is an ellipse whenever $A > C > B > 0$.

(b) Prove that the graph of the equation is a hyperbola if $AC < 0$ or $B < C < 4A < 0$.

(c) Prove that if the graph is a parabola, then $BC = 0$ or $A = B^2/(4C)$.

Proofs to Grade

11. Exercises throughout the text with this title ask you to examine "Proofs to Grade." These are allegedly true claims and supposed "proofs" of the claims. You should decide the merit of the claim and the validity of the proof and then assign a grade of

A (correct), if the claim and proof are correct, even if the proof is not the simplest or the proof you would have given.

C (partially correct), if the claim is correct *and* the proof is largely correct. The proof may contain one or two incorrect statements or justifications, but the errors are easily correctable.

F (failure), if the claim is incorrect, or the main idea of the proof is incorrect, or there are too many errors.

You must justify assignments of grades other than A and if the proof is incorrect, explain what is incorrect and why.

★ **(a)** Suppose a is an integer.

Claim. If a is odd then $a^2 + 1$ is even.

"Proof." Let a. Then, by squaring an odd we get an odd. An odd plus odd is even. So $a^2 + 1$ is even. ■

(b) Suppose a, b, and c are integers.

Claim. If a divides b and a divides c, then a divides $b + c$.

"Proof." Suppose a divides b and a divides c. Then for some integer q, $b = aq$, and for some integer q, $c = aq$. Then $b + c = aq + aq = 2aq = a(2q)$, so a divides $b + c$. ■

★ **(c)** Suppose x is a positive real number.

Claim. The sum of x and its reciprocal is greater than or equal to 2. That is,

$$x + \frac{1}{x} \geq 2.$$

"Proof." Multiplying by x, we get $x^2 + 1 \geq 2x$. By algebra, $x^2 - 2x + 1 \geq 0$. Thus, $(x - 1)^2 \geq 0$. Any real number squared is greater than or equal to zero, so $x + \frac{1}{x} \geq 2$ is true. ■

★ **(d)** Suppose m is an integer.

Claim. If m^2 is odd, then m is odd.

"Proof." Assume m is odd. Then $m = 2k + 1$ for some integer k. Therefore, $m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. Therefore, if m^2 is odd, then m is odd. ■

(e) Suppose a is an integer.

Claim. $a^3 + a^2$ is even.

"Proof." $a^3 + a^2 = a^2(a + 1)$, which is always an odd number times an even number. Therefore, $a^3 + a^2$ is even. ■

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