Proof by Contradiction. Suppose that the statement "If x and y are odd integers, then xy is odd" is false. Then x is odd and y is odd, and xy is not odd. Since xy is not odd, xy is even. Therefore 2 divides xy. Then by Euclid's Lemma, 2 divides x or 2 divides y. Thus either x is even or y is even. But x is odd and y is odd. This is a contradiction. We conclude that if x and y are odd integers, then xy is odd. ■

By now you may have the impression that, given a set of axioms and definitions of a mathematical system, any properly stated proposition in that system can be proved true or proved false. This is not the case. There are important examples in mathematics of **consistent axiom systems** (so that there exist structures satisfying all the axioms) for which there are statements such that neither the statement nor its negation can be proved. It is not a matter of these statements being difficult to prove or that no one has yet been clever enough to devise a proof; it has been proved that there can be no proof of either the statement or its negation within the system. Such statements are called **undecidable** in the system because their truth is independent of the truth of the axioms.

The classic case of an undecidable statement involves the fifth of five postulates that Euclid set forth as his basis for plane geometry: "Given a line and a point not on that line, exactly one line can be drawn through the point parallel to the line." For centuries, some thought Euclid's axioms were not independent, believing that the fifth postulate could be proved from the other four. It was not until the 19th century that it became clear that the fifth postulate was undecidable. There are now theories of Euclidean geometry for which the fifth postulate is assumed true and non-Euclidean geometries for which it is assumed false. Both are perfectly reasonable subjects for mathematical study and application.

Exercises 1.5

1. Analyze the logical form of each of the following statements and construct just the outline of a proof by the given method. Since the statements may contain terms with which you are not familiar, you should not (and perhaps could not) provide any details of the proof.

★ **(a)** Outline a proof by contraposition that if $(G, *)$ is a cyclic group, then $(G, *)$ is abelian.

(b) Outline a proof by contraposition that if $\mathbf{B}$ is a nonsingular matrix, then the determinant of $\mathbf{B}$ is not zero.

★ **(c)** Outline a proof by contradiction that the set of natural numbers is not finite.

(d) Outline a proof by contradiction that the multiplicative inverse of a nonzero real number x is unique.

★ **(e)** Outline a two-part proof that the inverse of the function f from A to B is a function from B to A if and only if f is one-to-one and onto B.

(f) Outline a two-part proof that a subset A of the real numbers is compact if and only if A is closed and bounded.

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2. A theorem of linear algebra states that if **A** and **B** are invertible matrices, then the product **AB** is invertible. As in Exercise 1,
- **(a)** outline a proof of the theorem by contraposition.
- **(b)** outline a proof of the converse of the theorem by contraposition.
- **(c)** outline a proof of the theorem by contradiction.
- **(d)** outline a proof of the converse of the theorem by contradiction.
- **(e)** outline a two-part proof that **A** and **B** are invertible matrices if and only if the product **AB** is invertible.

3. Let x, y, and z be integers. Write a proof by contraposition to show that
- ★ **(a)** if x is even, then $x + 1$ is odd.
- **(b)** if x is odd, then $x + 2$ is odd.
- **(c)** if x^2 is not divisible by 4, then x is odd.
- **(d)** if xy is even, then either x or y is even.
- ☆ **(e)** if $x + y$ is even, then x and y have the same parity.
- **(f)** if xy is odd, then both x and y are odd.
- **(g)** if 8 does not divide $x^2 - 1$, then x is even.
- **(h)** if x does not divide yz, then x does not divide z.

4. Write a proof by contraposition to show that for any real number x,
- **(a)** if $x^2 + 2x < 0$, then $x < 0$.
- ★ **(b)** if $x^2 - 5x + 6 < 0$, then $2 < x < 3$.
- **(c)** if $x^3 + x > 0$, then $x > 0$.

5. A circle has center (2, 4).
- **(a)** Prove that $(-1, 5)$ and (5, 1) are not both on the circle.
- **(b)** Prove that if the radius is less than 5, then the circle does not intersect the line $y = x - 6$.
- **(c)** Prove that if (0, 3) is not inside the circle, then (3, 1) is not inside the circle.

6. Suppose a and b are positive integers. Write a proof by contradiction to show that
- **(a)** if a divides b, then $a \leq b$.
- ★ **(b)** if ab is odd, then both a and b are odd.
- **(c)** if a is odd, then $a + 1$ is even.
- **(d)** if $a - b$ is odd, then $a + b$ is odd.
- **(e)** if $a < b$ and $ab < 3$, then $a = 1$.

7. Suppose a, b, c, and d are positive integers. Write a proof of each biconditional statement.
- **(a)** ac divides bc if and only if a divides b.
- **(b)** $a + 1$ divides b and b divides $b + 3$ if and only if $a = 2$ and $b = 3$.
- **(c)** a is odd if and only if $a + 1$ is even.
- **(d)** $a + c = b$ and $2b - a = d$ if and only if $a = b - c$ and $b + c = d$.

8. Let m and n be integers. Then prove that m and n have different parity iff $m^2 - n^2$ is odd.

9. Prove by contradiction that if n is a natural number, then

$$\frac{n}{n+1} > \frac{n}{n+2}.$$

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10. Prove that $\sqrt{5}$ is not a rational number.

11. Three real numbers, x, y, and z, are chosen between 0 and 1 with $0 < x < y < z < 1$. Prove that at least two of the numbers x, y, and z are within $\frac{1}{2}$ unit from one another.

Proofs to Grade

12. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) Suppose m is an integer.

Claim. If m^2 is odd, then m is odd.

"Proof." Assume that m^2 is not odd. Then m^2 is even and $m^2 = 2k$ for some integer k. Thus $2k$ is a perfect square; that is, $\sqrt{2k}$ is an integer. If $\sqrt{2k}$ is odd, then $\sqrt{2k} = 2n + 1$ for some integer n, which means $m^2 = 2k = (2n + 1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$. Thus m^2 is odd, contrary to our assumption. Therefore $\sqrt{2k} = m$ must be even. Thus if m^2 is not odd, then m is not odd. Hence if m^2 is odd, then m is odd. ■

★ **(b)** Suppose t is a real number.

Claim. If t is irrational, then $5t$ is irrational.

"Proof." Suppose $5t$ is rational. Then $5t = p/q$, where p and q are integers and $q \neq 0$. Therefore, $t = p/(5q)$, where p and $5q$ are integers and $5q \neq 0$, so t is rational. Therefore, if t is irrational, then $5t$ is irrational. ■

(c) Suppose x and y are integers.

Claim. If x and y are even then $x + y$ is even.

"Proof." Suppose x and y are even but $x + y$ is odd. Then, for some integer k, $x + y = 2k + 1$. Therefore, $x + y + (-2)k = 1$. The left side of the equation is even because it is the sum of even numbers. However, the right side, 1, is odd. Since an even cannot equal an odd, we have a contradiction. Therefore, $x + y$ is even. ■

(d) Suppose a, b, and c are integers.

Claim. If a divides both b and c, then a divides $b + c$.

"Proof." Assume that a does not divide $b + c$. Then there is no integer k such that $ak = b + c$. However, a divides b, so $am = b$ for some integer m; and a divides c, so $an = c$ for some integer n. Thus $am + an = a(m + n) = b + c$. Therefore $k = m + n$ is an integer satisfying $ak = b + c$. Thus the assumption that a does not divide $b + c$ is false, and a does divide $b + c$. ■

1.6 Proofs Involving Quantifiers

Recall that in our first example of a direct proof in Section 1.4 we proved the statement "If x is odd then $x + 1$ is even." That statement has the meaning "For every integer x, if x is odd then $x + 1$ is even." We dealt with the quantifier in that example by asking you to think of the variable x as some fixed integer. This section discusses specifically the proof methods for statements with quantifiers.

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