Exercises 1.6

1. Prove that

★ **(a)** there exist integers m and n such that $2m + 7n = 1$.

(b) there exist integers m and n such that $15m + 12n = 3$.

★ **(c)** there do not exist integers m and n such that $2m + 4n = 7$.

(d) there do not exist integers m and n such that $12m + 15n = 1$.

(e) for every integer t, if there exist integers m and n such that $15m + 16n = t$, then there exist integers r and s such that $3r + 8s = t$.

☆ **(f)** if there exist integers m and n such that $12m + 15n = 1$, then m and n are both positive.

(g) for every odd integer m, if m has the form $4k + 1$ for some integer k, then $m + 2$ has the form $4j - 1$ for some integer j.

(h) for every odd integer m, $m^2 = 8k + 1$ for some integer k. (*Hint:* Use the fact that $k(k + 1)$ is an even integer for every integer k.)

(i) for all odd integers m and n, if $mn = 4k - 1$ for some integer k, then m or n is of the form $4j - 1$ for some integer j.

2. Prove that for all integers a, b, and c,

(a) if c divides a and c divides b, then for all integers x and y, c divides $ax + by$.

★ **(b)** if a divides $b - 1$ and a divides $c - 1$, then a divides $bc - 1$.

(c) if a divides b, then for all natural numbers n, a^n divides b^n.

(d) if a is odd, $c > 0$, c divides a and c divides $a + 2$, then $c = 1$.

(e) if there exist integers m and n such that $am + bn = 1$ and $c \neq \pm 1$, then c does not divide a or c does not divide b.

3. Prove that if every even natural number greater than 2 is the sum of two primes,* then every odd natural number greater than 5 is the sum of three primes.

4. Provide either a proof or a counterexample for each of these statements.

(a) For all positive integers x, $x^2 + x + 41$ is a prime.

(b) $(\forall x)(\exists y)(x + y = 0)$. (Universe of all reals)

(c) $(\forall x)(\forall y)(x > 1 \wedge y > 0 \Rightarrow y^x > x)$. (Universe of all reals)

(d) For integers a, b, c, if a divides bc, then either a divides b or a divides c.

(e) For integers a, b, c, and d, if a divides $b - c$ and a divides $c - d$, then a divides $b - d$.

(f) For all positive real numbers x, $x^2 - x \geq 0$.

(g) For all positive real numbers x, $2^x > x + 1$.

☆ **(h)** For every positive real number x, there is a positive real number y less than x with the property that for all positive real numbers z, $yz \geq z$.

☆ **(i)** For every positive real number x, there is a positive real number y with the property that if $y < x$, then for all positive real numbers z, $yz \geq z$.

5. **(a)** Prove that the natural number x is prime iff $x > 1$ and there is no positive integer greater than 1 and less than or equal to $\sqrt{x}$ that divides x.

* No one knows whether every even number greater than 2 is the sum of two prime numbers. This is the famous Goldbach Conjecture, proposed by the Prussian mathematician Christian Goldbach in 1742. You should search the Web to learn about the million dollar prize (never claimed) for proving Goldbach's Conjecture. Fortunately, you don't have to prove Goldbach's Conjecture to do this exercise.

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(b) Prove that if p is a prime number and $p \neq 3$, then 3 divides $p^2 + 2$. (*Hint:* When p is divided by 3, the remainder is either 0, 1, or 2. That is, for some integer k, $p = 3k$ or $p = 3k + 1$ or $p = 3k + 2$.)

6. Prove that

(a) for every natural number n, $\frac{1}{n} \leq 1$. (*Hint:* Use the fact that $n \geq 1$ and divide by the positive number n.)

(b) there is a natural number M such that for all natural numbers $n > M$, $\frac{1}{n} < 0.13$.

★ **(c)** for every natural number n, there is a natural number M such that $2n < M$.

(d) there is a natural number M such that for every natural number n, $\frac{1}{n} < M$.

(e) there is no largest natural number.

(f) there is no smallest positive real number.

★ **(g)** for every real number $\varepsilon > 0$, there is a natural number M such that for all natural numbers $n > M$, $\frac{1}{n} < \varepsilon$.

☆ **(h)** for every real number $\varepsilon > 0$, there is a natural number M such that if $m > n > M$, then $\frac{1}{n} - \frac{1}{m} < \varepsilon$.

(i) there is a natural number K such that $\frac{1}{r^2} < 0.01$ whenever r is a real number larger than K.

(j) there exist integers L and G such that $L < G$ and for every real number x, if $L < x < G$, then $40 > 10 - 2x > 12$.

(k) there exists an odd integer M such that for all real numbers r larger than M, $\frac{1}{2r} < 0.01$.

(l) for every natural number x there is an integer k such that $3.3x + k < 50$.

(m) there exist integers $x < 100$ and $y < 30$ such that $x + y < 128$ and for all real numbers r and s, if $r > x$ and $s > y$, then $(r - 50)(s - 20) > 390$.

(n) for every pair of positive real numbers x and y where $x < y$, there exists a natural number M such that if n is a natural number and $n > M$, then $\frac{1}{n} < (y - x)$.

Proofs to Grade

7. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** Every polynomial of degree 3 with real coefficients has a real zero.
"Proof." The polynomial $p(x) = x^3 - 8$ has degree 3, real coefficients, and a real zero ($x = 2$). Thus the statement "Every polynomial of degree 3 with real coefficients does not have a real zero" is false, and hence its denial, "Every polynomial of degree 3 with real coefficients has a real zero," is true. ■

★ **(b)** **Claim.** There is a unique polynomial whose first derivative is $2x + 3$ and which has a zero at $x = 1$.
"Proof." The antiderivative of $2x + 3$ is $x^2 + 3x + C$. If we let $p(x) = x^2 + 3x - 4$, then $p'(x) = 2x + 3$ and $p(1) = 0$. So $p(x)$ is the desired polynomial. ■

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(c) **Claim.** Every prime number greater than 2 is odd.
"Proof." The prime numbers greater than 2 are 3, 5, 7, 11, 13, 17, 19, ... None of these are even, so all of them are odd. ■

★ **(d)** **Claim.** There exists an irrational number r such that $r^{\sqrt{2}}$ is rational.
"Proof." If $\sqrt{3}^{\sqrt{2}}$ is rational, then $r = \sqrt{3}$ is the desired example. Otherwise, $\sqrt{3}^{\sqrt{2}}$ is irrational and $(\sqrt{3}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{3})^2 = 3$, which is rational. Therefore either $\sqrt{3}$ or $\sqrt{3}^{\sqrt{2}}$ is an irrational number r such that $r^{\sqrt{2}}$ is rational. ■

(e) **Claim.** For every real number x, $|x| \geq 0$.
"Proof." We proceed by three cases: $x > 0$, $x = 0$, and $x < 0$.

Case 1. $x > 0$. Choose, for example, $x = 4$. Then $|4| = 4$. Thus $|x| \geq 0$.
Case 2. $x = 0$. Then $|0| = 0$. Thus, $|x| \geq 0$.
Case 3. $x < 0$. Choose, for example, $x = -5$. Then $|-5| = 5$. Thus $|x| \geq 0$. ■

(f) **Claim.** If x is prime, then $x + 7$ is composite.
"Proof." Let x be a prime number. If $x = 2$, then $x + 7 = 9$, which is composite. If $x \neq 2$, then x is odd, so $x + 7$ is even and greater than 2. In this case, too, $x + 7$ is composite. Therefore, if x is prime, then $x + 7$ is composite. ■

(g) **Claim.** For all irrational numbers t, $t - 8$ is irrational.
"Proof." Suppose there exists an irrational number t such that $t - 8$ is rational. Then $t - 8 = \frac{p}{q}$, where p and q are integers and $q \neq 0$. Then $t = \frac{p}{q} + 8 = \frac{p + 8q}{q}$, with $p + 8q$ and q integers and $q \neq 0$. This is a contradiction because t is irrational. Therefore, for all irrational numbers t, $t - 8$ is irrational. ■

(h) **Claim.** For real numbers x and y, if $xy = 0$ then $x = 0$ or $y = 0$.
"Proof."

Case 1. If $x = 0$, then $xy = 0y = 0$.
Case 2. If $y = 0$, then $xy = x0 = 0$.

In either case, $xy = 0$. ■

☆ **(i)** **Claim.** For every real number $\varepsilon > 0$, there is a natural number K such that for all real numbers $x > K$, $\frac{1}{4x} < \varepsilon$.

"Proof." Let $\varepsilon > 0$ be a real number. Let K be $\frac{1}{2\varepsilon}$. Assume x is a real number and $x > K$. Then $x > \frac{1}{2\varepsilon}$, so $x > \frac{1}{4\varepsilon}$. Therefore, $4x\varepsilon > 1$, so $\frac{1}{4x} < \varepsilon$. ■

(j) **Claim.** For every natural number n, $n \leq n^2$.
"Proof." Let n be a natural number. Since n is a natural number, $1 \leq n$. Since n is positive, $n \cdot 1 \leq n \cdot n$. Therefore, $n \leq n^2$ for all natural numbers n. ■

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