The second half of the proof of Theorem 2.1.5 could have been done differently. We could have shown that $A \subseteq B$ by giving a direct proof that $x \in A$ implies $x \in B$. A proof that consists of a series of steps beginning with "Assume $x \in A$" and leads to a conclusion that "Therefore $x \in B$" is often called an *element-chasing proof*, and is the natural way to prove the most basic facts about sets. As we build our knowledge of set properties in the next section, we may use theorems already proved, as we did above by using Theorem 2.1.1, to shorten our proof of Theorem 2.1.5. An element-chasing proof of part (ii) would be just as correct, but most people prefer a shorter, more elegant proof. When you write proofs, you may choose one method of proof over another because it is shorter, is easier to understand, or for any other reason.

Exercises 2.1

1. Write the following sets by using the set notation $\{x\colon P(x)\}$.

★ **(a)** The set of natural numbers strictly less than 6

(b) The set of integers whose square is less than 17

★ **(c)** $[2, 6]$

(d) $(-1, 9]$

(e) $[-5, -1)$

(f) The set of rational numbers less than -1

2. Let $X = \{x\colon P(x)\}$. Are the following statements true or false?

(a) If $a \in X$, then $P(a)$.

(b) If $P(a)$, then $a \in X$.

(c) If $\sim P(a)$, then $a \notin X$.

3. ★ **(a)** (*Russell paradox*) A logical difficulty arises from the idea, which at first appears natural, of calling any collection of objects a set. Let's say that set B is **ordinary** if $B \notin B$. For example, if B is the set of all chairs, then $B \notin B$, because B is not a chair. It is only in the case of very unusual collections that we are tempted to say that a set is a member of itself. (The collection of all abstract ideas certainly is an abstract idea.) Let $X = \{x\colon x \text{ is an ordinary set}\}$. Is $X \in X$? Is $X \notin X$? What should we say about the collection of all ordinary sets?

(b) In the town of Seville, the (male) barber shaves all the men, and only the men, who do not shave themselves. Let A be the set of all men in the town who do not shave themselves. Who shaves the barber? (That is, is the barber an element of A? Is he not an element of A?)

4. True or false?

★ **(a)** $\mathbb{N} \subseteq \mathbb{Q}$.

(b) $\mathbb{Q} \subseteq \mathbb{Z}$.

★ **(c)** $\mathbb{N} \subseteq \mathbb{R}$.

(d) $\left[\frac{1}{2}, \frac{5}{2}\right] \subseteq \mathbb{Q}$.

★ **(e)** $\left[\frac{1}{2}, \frac{5}{2}\right] \subseteq \left(\frac{1}{2}, \frac{5}{2}\right)$.

(f) $\mathbb{R} \subseteq \mathbb{Q}$.

★ **(g)** $[7, 10] \subseteq \mathbb{R}$.

(h) $[2, 5] = \{2, 3, 4, 5\}$.

★ **(i)** $[7, 10) \subseteq \{7, 8, 9, 10\}$.

(j) $(6, 9] \subseteq [6, 10)$.

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5. True or false?

★ **(a)** $\varnothing \in \{\varnothing, \{\varnothing\}\}$. **(b)** $\varnothing \subseteq \{\varnothing, \{\varnothing\}\}$.

★ **(c)** $\{\varnothing\} \in \{\varnothing, \{\varnothing\}\}$. **(d)** $\{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$.

★ **(e)** $\{\{\varnothing\}\} \in \{\varnothing, \{\varnothing\}\}$. **(f)** $\{\{\varnothing\}\} \subseteq \{\varnothing, \{\varnothing\}\}$.

★ **(g)** For every set A, $\varnothing \in A$. **(h)** For every set A, $\{\varnothing\} \subseteq A$.

★ **(i)** $\{\varnothing, \{\varnothing\}\} \subseteq \{\{\varnothing, \{\varnothing\}\}\}$. **(j)** $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$.

★ **(k)** $\{1, 2, 3\} \subseteq \{1, 2, 3, \{4\}\}$. **(l)** $\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\}$.

6. Give an example, if there is one, of sets A, B, and C such that the following are true. If there is no example, write "Not possible."

★ **(a)** $A \subseteq B$, $B \nsubseteq C$, and $A \subseteq C$.

(b) $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$.

★ **(c)** $A \nsubseteq B$, $B \nsubseteq C$, and $A \subseteq C$.

(d) $A \subseteq B$, $B \nsubseteq C$, and $A \nsubseteq C$.

7. Prove that if $x \notin B$ and $A \subseteq B$, then $x \notin A$.

☆ **8.** Prove part (c) of Theorem 2.1.1: For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

☆ **9.** Prove that if $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$, then $A = B$ and $B = C$.

10. Suppose that $X = \{x: x \in \mathbb{R} \text{ and } x \text{ is a solution to } x^2 - 7x + 12 = 0\}$ and $Y = \{3, 4\}$. Prove that $X = Y$.

11. Let $X = \{x \in \mathbb{Z}: |x| \leq 3\}$ and $Y = \{-3, -2, -1, 0, 1, 2, 3\}$. Prove that $X = Y$.

12. Prove that $X = Y$, where $X = \{x \in \mathbb{N}: x^2 < 30\}$ and $Y = \{1, 2, 3, 4, 5\}$.

13. For a natural number a, let $a\mathbb{Z}$ be the set of all integer multiples of a. Prove that for all $a, b \in \mathbb{N}$, $a = b$ iff $a\mathbb{Z} = b\mathbb{Z}$.

14. Write the power set, $\mathcal{P}(X)$, for each of the following sets.

★ **(a)** $X = \{0, \triangle, \square\}$ **(b)** $X = \{S, \{S\}\}$

★ **(c)** $X = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ **(d)** $X = \{1, \{\varnothing\}, \{2, \{3\}\}\}$

15. Let A, B, and C be sets and x and y be any objects. True or false?

★ **(a)** If $x \in A$, then $x \in \mathcal{P}(A)$.

(b) If $x \in A$, then $\{x\} \in \mathcal{P}(A)$.

(c) If $x \in A$, then $\{x\} \subseteq \mathcal{P}(A)$.

(d) If $\{x, y\} \in \mathcal{P}(A)$, then $x \in A$ and $y \in A$.

★ **(e)** If $B \subseteq A$, then $\{B\} \in \mathcal{P}(A)$.

(f) If $B \subseteq A$, then $B \in \mathcal{P}(A)$.

★ **(g)** If $B \in \mathcal{P}(A)$, then $B \subseteq A$.

(h) If $C \subseteq B$ and $B \in \mathcal{P}(A)$, then $C \in \mathcal{P}(A)$.

16. List all of the proper subsets for each of the following sets.

★ **(a)** $\varnothing$ **(b)** $\{\varnothing, \{\varnothing\}\}$

★ **(c)** $\{1, 2\}$ **(d)** $\{0, \triangle, \square\}$

17. True or false?

★ **(a)** $\varnothing \in \mathcal{P}(\{\varnothing, \{\varnothing\}\})$. **(b)** $\{\varnothing\} \in \mathcal{P}(\{\varnothing, \{\varnothing\}\})$.

★ **(c)** $\{\{\varnothing\}\} \in \mathcal{P}(\{\varnothing, \{\varnothing\}\})$. **(d)** $\varnothing \subseteq \mathcal{P}(\{\varnothing, \{\varnothing\}\})$.

★ **(e)** $\{\varnothing\} \subseteq \mathcal{P}(\{\varnothing, \{\varnothing\}\})$. **(f)** $\{\{\varnothing\}\} \subseteq \mathcal{P}(\{\varnothing, \{\varnothing\}\})$.

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★ **(g)** $3 \in \mathbb{Q}$. **(h)** $\{3\} \subseteq \mathcal{P}(\mathbb{Q})$.
★ **(i)** $\{3\} \in \mathcal{P}(\mathbb{Q})$. **(j)** $\{\{3\}\} \subseteq \mathcal{P}(\mathbb{Q})$.
★ **(k)** $\{3\} \subseteq \mathbb{Q}$. **(l)** $\{\{3\}\} \in \mathcal{P}(\mathbb{Q})$.

18. Let A and B be sets. Prove that $A = B$ iff $\mathcal{P}(A) = \mathcal{P}(B)$.

Proofs to Grade

19. Assign a grade A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) **Claim.** If $X = \{x \in \mathbb{N}: x^2 < 14\}$ and $Y = \{1, 2, 3\}$, then $X = Y$.
"Proof." Since $1^2 = 1 < 14$, $2^2 = 4 < 14$, and $3^2 = 9 < 14$, $X = Y$. ■

(b) **Claim.** If A, B, and C are sets, and $A \subseteq B$ and $B \subseteq C$, the $A \subseteq C$.
"Proof." Let $A = \{1, 5, 8\}$, $B = \{1, 4, 5, 8, 10\}$, and $C = \{1, 2, 4, 5, 6, 8, 10\}$. Then $A \subseteq B$, and $B \subseteq C$, and $A \subseteq C$. ■

★ **(c)** **Claim.** If A, B, and C are sets, and $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
"Proof." Suppose x is any object. If $x \in A$, then $x \in B$, since $A \subseteq B$. If $x \in B$, then $x \in C$, since $B \subseteq C$. Thus, $x \in C$. Therefore, $A \subseteq C$. ■

(d) **Claim.** If A, B, and C are sets, and $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
"Proof." If $x \in C$, then, since $B \subseteq C$, $x \in B$. Since $A \subseteq B$ and $x \in B$, it follows that $x \in A$. Thus $x \in C$ implies $x \in A$. Therefore, $A \subseteq C$. ■

★ **(e)** **Claim.** If A, B, and C are sets, and $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
"Proof." Suppose $A \subseteq B$ and $B \subseteq C$. Then $x \in A$ and $x \in B$, because $A \subseteq B$. Then $x \in B$ and $x \in C$, because $B \subseteq C$. Therefore, $x \in A$ and $x \in C$, so $A \subseteq C$. ■

(f) **Claim.** If A is a set, $A \subseteq \mathcal{P}(A)$.
"Proof." Assume A is a set. Suppose $x \in A$. Then $x \subseteq A$. Thus $x \in \mathcal{P}(A)$. Therefore $A \subseteq \mathcal{P}(A)$. ■

(g) **Claim.** If A is a set, $A \subseteq \mathcal{P}(A)$.
"Proof." Assume A is a set. Suppose $x \in A$. Then $\{x\} \subseteq A$. Thus $\{x\} \in \mathcal{P}(A)$. Therefore, $A \subseteq \mathcal{P}(A)$. ■

★ **(h)** **Claim.** If A and B are sets and $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.
"Proof." $x \in A \Rightarrow \{x\} \subseteq A$
$\Rightarrow \{x\} \in \mathcal{P}(A)$
$\Rightarrow \{x\} \in \mathcal{P}(B)$
$\Rightarrow \{x\} \subseteq B$
$\Rightarrow x \in B$.
Therefore, $x \in A \Rightarrow x \in B$. Thus $A \subseteq B$. ■

★ **(i)** **Claim.** If $A \subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.
"Proof." Suppose $A \subseteq B$ and $B \not\subseteq C$. Then there exists $x \in B$ such that $x \notin C$. Since $x \in B$, $x \in A$ by definition of subset. Thus $x \in A$ and $x \notin C$. Therefore $A \not\subseteq C$. ■

2.2 Set Operations

In this section we give precise definitions and prove some well-known properties of familiar operations on sets. Set union, interesection, and difference are called *binary* operations because each combines two sets to produce another set.

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