If $(x, y) \in C \times D$, a similar argument shows $(x, y) \in (A \cup C) \times (B \cup D)$. This shows that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. ■

Parts (b), (c), (d), and (f) are proved in Exercise 15. Part (e) of Theorem 2.2.3 cannot be sharpened to equality. See Exercise 16(a).

Exercises 2.2

1. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{0, 2, 4, 6, 8\}$, $C = \{1, 2, 4, 5, 7, 8\}$, and $D = \{1, 2, 3, 5, 6, 7, 8, 9, 10\}$. Find

★ **(a)** $A \cup B$. **(b)** $A \cap B$.
★ **(c)** $A - B$. **(d)** $A - (B - C)$.
★ **(e)** $(A - B) - C$. **(f)** $A \cup (C \cap D)$.
★ **(g)** $(A \cap C) \cap D$. **(h)** $A \cap (B \cup C)$.
★ **(i)** $(A \cap B) \cup (A \cap C)$. **(j)** $(A \cup B) - (C \cap D)$.

2. Let the universe be all real numbers. Let $A = [3, 8)$, $B = [2, 6]$, $C = (1, 4)$, and $D = (5, \infty)$. Find

(a) $A \cup B$. ★ **(b)** $A \cup C$.
(c) $A \cap B$. ★ **(d)** $B \cap C$.
(e) $A - B$. **(f)** $B - D$.
(g) $D - A$. ★ **(h)** A^c.
(i) $B - (A \cup C)$. **(j)** $(A \cup C) - (B \cap D)$.

3. Let the universe be the set $\mathbb{Z}$. Let E, D, $\mathbb{Z}^+$, and $\mathbb{Z}^-$ be the sets of all even, odd, positive, and negative integers, respectively. Find

★ **(a)** $E - \mathbb{Z}^+$. **(b)** $\mathbb{Z}^+ - E$. **(c)** $D - E$.
★ **(d)** $(\mathbb{Z}^+)^c$. **(e)** $\mathbb{Z}^+ - \mathbb{Z}^-$. **(f)** E^c.
★ **(g)** $E - \mathbb{Z}^-$. **(h)** $(E \cap \mathbb{Z}^-)^c$. **(i)** $\varnothing^c$.

★ **4.** Let A, B, C, and D be as in Exercise 1. Which pairs of these four sets are disjoint?

5. Let A, B, C, and D be as in Exercise 2. Which pairs of these four sets are disjoint?

6. Give an example of nonempty sets A, B and C such that

★ **(a)** $C \subseteq A \cup B$ and $A \cap B \nsubseteq C$.
(b) $A \subseteq B$ and $C \subseteq A \cap B$.
(c) $A \cup B \subseteq C$ and $C \nsubseteq B$.
(d) $A \nsubseteq B \cup C$, $B \nsubseteq A \cup C$, and $C \subseteq A \cup B$.
(e) $A \subseteq B \cup C$, $B \subseteq A \cup C$, $C \subseteq A \cup B$, $A \cap B = A \cap C$, and $A \neq B$.
(f) $A \cap B \subseteq C$, $A \cap C \subseteq B$, $B \cap C \subseteq A$, and $A = B \cup C$.

7. Prove the remaining parts of Theorem 2.2.1.

8. Prove the remaining parts of Theorem 2.2.2.

9. Let A, B, and C be sets. Prove that

☆ **(a)** $A \subseteq B$ iff $A - B = \varnothing$.

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(b) if $A \subseteq B \cup C$ and $A \cap B = \emptyset$, then $A \subseteq C$.
★ **(c)** $C \subseteq A \cap B$ iff $C \subseteq A$ and $C \subseteq B$.
(d) if $A \subseteq B$, then $A - C \subseteq B - C$.
(e) $(A - B) - C = (A - C) - (B - C)$.
(f) if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.
(g) $(A \cup B) \cap C \subseteq A \cup (B \cap C)$.
(h) $A - B$ and B are disjoint.

10. Let A, B, C, and D be sets. Prove that
(a) if $C \subseteq A$ and $D \subseteq B$, then $C \cap D \subseteq A \cap B$.
(b) if $C \subseteq A$ and $D \subseteq B$, then $C \cup D \subseteq A \cup B$.
★ **(c)** if $C \subseteq A$, $D \subseteq B$, and A and B are disjoint, then C and D are disjoint.
(d) if $C \subseteq A$ and $D \subseteq B$, then $D - A \subseteq B - C$.
(e) if $A \cup B \subseteq C \cup D$, $A \cap B = \emptyset$, and $C \subseteq A$, then $B \subseteq D$.

11. Provide counterexamples for each of the following.
★ **(a)** If $A \cup C \subseteq B \cup C$, then $A \subseteq B$.
(b) If $A \cap C \subseteq B \cap C$, then $A \subseteq B$.
★ **(c)** If $(A - B) \cap (A - C) = \emptyset$, then $B \cap C = \emptyset$.
(d) $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.
★ **(e)** $A - (B - C) = (A - B) - (A - C)$.
(f) $A - (B - C) = (A - B) - C$.

12. Let A and B be sets.
★ **(a)** Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. You may use Exercise 9(c).
(b) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
(c) Show by example that set equality need not be the case in part (b). Under what conditions on A and B is $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$?
★ **(d)** Show that there are no sets A and B such that $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

13. List the ordered pairs in $A \times B$ and $B \times A$ in each case:
(a) $A = \{1, 3, 5\}$, $B = \{a, e, k, n, r\}$.
★ **(b)** $A = \{1, 2, \{1, 2\}\}$, $B = \{q, \{t\}, \pi\}$.
(c) $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, $B = \{(\emptyset, \{\emptyset\}), \{\emptyset\}, (\{\emptyset\}, \emptyset)\}$.
(d) $A = \{(2, 4), (3, 1)\}$, $B = \{(4, 1), (2, 3)\}$.

14. Let A and B be nonempty sets. Prove that $A \times B = B \times A$ iff $A = B$.

15. Complete the proof of Theorem 2.2.3 by proving
★ **(a)** $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
(b) $A \times \emptyset = \emptyset$.
(c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(d) $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

16. Give an example of nonempty sets A, B, C, and D such that
(a) $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$.
(b) $(C \times C) - (A \times B) \neq (C - A) \times (C - B)$.
(c) $A \times (B \times C) \neq (A \times B) \times C$.

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17. One way to define an ordered pair in terms of sets is to say $(a, b) = \{\{a\}, \{a, b\}\}$. Using this definition, prove that $(a, b) = (x, y)$ iff $a = x$ and $b = y$.

18. Let A and B be sets. Define the **symmetric difference** of A and B to be $A \Delta B = (A - B) \cup (B - A)$. Prove that

(a) $A \Delta B = B \Delta A$. **(b)** $A \Delta B = (A \cup B) - (A \cap B)$.

(c) $A \Delta A = \varnothing$. **(d)** $A \Delta \varnothing = A$.

Proofs to Grade

19. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** If $A \subseteq B$, then $A - C \subseteq B - C$.

"Proof." Assume $A \subseteq B$. Suppose $x \in A$. Then $x \in B$, since $A \subseteq B$. Let C be any set. Then $x \in A$ and $x \notin C$. Then $x \in B$ and $x \notin C$. Thus $x \in A - C$ and $x \in B - C$. Therefore, $A - C \subseteq B - C$. ■

☆ **(b)** **Claim.** If $A \subseteq B$, then $A - C \subseteq B - C$.

"Proof." Assume $A \subseteq B$. Suppose $A - C$. Then $x \in A$ and $x \notin C$. Then $x \in B$, because $A \subseteq B$. Since $x \in B$ and $x \notin C$, $B - C$. Therefore, $A - C \subseteq B - C$. ■

(c) **Claim.** If $A \subseteq B$, then $A - C \subseteq B - C$.

"Proof." Assume $A \subseteq B$. Then $x \in A$ and $x \in B$. Suppose $x \in A - C$. Then $x \in A$ and $x \notin C$. Since $x \in B$ and $x \notin C$, $x \in B - C$. Therefore, $A - C \subseteq B - C$. ■

★ **(d)** **Claim.** $A \subseteq B$ iff $A \cap B = A$.

"Proof." Assume that $A \subseteq B$. Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $x \in A$. This shows that $A \cap B = A$. Now assume that $A \cap B = A$. Suppose $x \in A$. Then $x \in A \cap B$, since $A = A \cap B$; and, therefore, $x \in B$. This shows that $x \in A$ implies $x \in B$, and so $A \subseteq B$. ■

★ **(e)** **Claim.** $A \cap \varnothing = A$.

"Proof." We know that $x \in A \cap \varnothing$ iff $x \in A$ and $x \in \varnothing$. Since $x \in \varnothing$ is false, $x \in A$ and $x \in \varnothing$ iff $x \in A$. Therefore, $x \in A \cap \varnothing$ iff $x \in A$; that is, $A \cap \varnothing = A$. ■

(f) **Claim.** If $A \cap B \neq \varnothing$ and $B \cap C \neq \varnothing$, then $A \cap C \neq \varnothing$.

"Proof." Assume $A \cap B \neq \varnothing$ and $B \cap C \neq \varnothing$. Since $A \cap B \neq \varnothing$, there exists x such that $x \in A \cap B$; thus $x \in A$. Since $B \cap C \neq \varnothing$, there exists $x \in B \cap C$; thus $x \in C$. Hence $x \in A$ and $x \in C$. Therefore, $x \in A \cap C$, which show $A \cap C \neq \varnothing$. ■

(g) **Claim.** $A \cap A^c = \varnothing$.

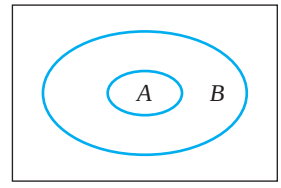
"Proof." ⟨*We show each side is a subset of the other.*⟩ By Theorem 2.1.1, $\varnothing \subseteq A \cap A^c$. Now suppose $x \in A \cap A^c$. Then $x \in A$ and $x \in A^c$. Thus $x \in A$ and $x \notin A$. Therefore, $x \neq x$. Hence, by the definition of $\varnothing$, $x \in \varnothing$. Therefore, $A \cap A^c \subseteq \varnothing$. ■

(h) **Claim.** $\mathcal{P}(A - B) - \{\varnothing\} \subseteq \mathcal{P}(A) - \mathcal{P}(B)$.

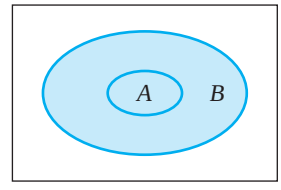
"Proof." Suppose $x \in \mathcal{P}(A - B) - \{\varnothing\}$. Then $x \in \mathcal{P}(A - B)$ and $x \neq \varnothing$. Since $\varnothing \in \mathcal{P}(A)$ and $\varnothing \in \mathcal{P}(B)$, $\varnothing \notin \mathcal{P}(A) - \mathcal{P}(B)$. Since $x \in \mathcal{P}(A - B)$, $x \in \mathcal{P}(A) - \mathcal{P}(B)$. Therefore, we can conclude that $\mathcal{P}(A - B) - \{\varnothing\} \subseteq \mathcal{P}(A) - \mathcal{P}(B)$. ■

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★ **(i)** **Claim.** If $A \subseteq B$, then $A \cup B = B$.
"Proof." Let $A \subseteq B$. Then A and B are related as in this figure.



Since $A \cup B$ is the set of elements in either of the sets A or B, $A \cup B$ is the shaded area in this figure.



Since this is B, $A \cup B = B$. ■

2.3 Extended Set Operations and Indexed Families of Sets

A set of sets is often called a **family** or a **collection** of sets. In this section we extend the definitions of union and intersection to families of sets and prove generalizations of parts of Theorem 2.2.1.

Throughout this section we will use script letters, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, . . . to denote families of sets. For example,

$$\mathcal{A} = \{\{1, 2, 3\}, \{3, 4, 5\}, \{3, 6\}, \{2, 3, 6, 7, 9, 10\}\}$$

is a family consisting of four sets. Notice that $5 \in \{3, 4, 5\}$ and $\{3, 4, 5\} \in \mathcal{A}$, but $5 \notin \mathcal{A}$. The set $\mathcal{B} = \{(-x, x): x \in \mathbb{R} \text{ and } x > 0\}$ is a family of open intervals. The sets $(-1, 1)$, $(-\sqrt{2}, \sqrt{2})$, and $(-5, 5)$ are elements of $\mathcal{B}$. See Figure 2.3.1.

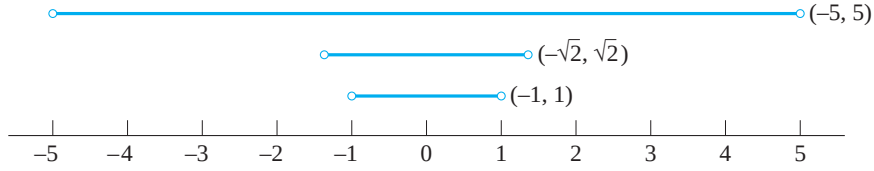


Figure 2.3.1

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