Exercises 2.3

1. Find the union and intersection of each of the following families or indexed collections.

★ **(a)** Let $\mathscr{A} = \{\{1, 2, 3, 4, 5\}, \{2, 3, 4, 5, 6\}, \{3, 4, 5, 6, 7\}, \{4, 5, 6, 7, 8\}\}$.

(b) Let $\mathscr{A} = \{\{1, 3, 5\}, \{2, 4, 6\}, \{7, 9, 11, 13\}, \{8, 10, 12\}\}$.

★ **(c)** For each natural number n, let $A_n = \{5n, 5n + 1, 5n + 2 \ldots, 6n\}$, and let $\mathscr{A} = \{A_n : n \in \mathbb{N}\}$.

(d) For each natural number n, let $B_n = \mathbb{N} - \{1, 2, 3, \ldots, n\}$ and let $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$.

★ **(e)** Let $\mathscr{A}$ be the set of all sets of integers that contain 10.

(f) Let $A_1 = \{1\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 4, 5\}$, $\ldots$, $A_{10} = \{10, 11, \ldots, 19\}$, and let $\mathscr{A} = \{A_n : n \in \{1, 2, 3, \ldots, 10\}\}$.

★ **(g)** For each natural number, let $A_n = \left(0, \frac{1}{n}\right)$, and let $\mathscr{A} = \{A_n : n \in \mathbb{N}\}$.

(h) For $r \in (0, \infty)$, let $A_r = [-\pi, r)$, and let $\mathscr{A} = \{A_r : r \in (0, \infty)\}$.

★ **(i)** For each real number r, let $A_r = [|r|, 2|r| + 1]$, and let $\mathscr{A} = \{A_r : r \in \mathbb{R}\}$.

(j) For each $n \in \mathbb{N}$, let $M_n = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$, and let $\mathscr{M} = \{M_n : n \in \mathbb{N}\}$.

(k) For each natural number $n \geq 3$, let $A_n = \left[\frac{1}{n}, 2 + \frac{1}{n}\right]$ and $\mathscr{A} = \{A_n : n \geq 3\}$.

(l) For each $n \in \mathbb{Z}$, let $C_n = [n, n + 1)$ and let $\mathscr{C} = \{C_n : n \in \mathbb{Z}\}$.

(m) For each $n \in \mathbb{Z}$, let $A_n = (n, n + 1)$ and $\mathscr{A} = \{A_n : n \in \mathbb{Z}\}$.

(n) For each $n \in \mathbb{Z}$, let $D_n = \left(-n, \frac{1}{n}\right)$ and $\mathscr{D} = \{D_n : n \in \mathbb{N}\}$.

(o) For each prime number p, let $p\mathbb{N} = \{np : n \in \mathbb{N}\}$ and $\mathscr{A}$ be the family $\{p\mathbb{N} : n \in \mathbb{N}$ and p is prime$\}$.

★ **(p)** For each $n \in \mathbb{Z}$, let $T_n = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq 1, 0 \leq y \leq x^n\}$ and $\mathscr{T} = \{T_n : n \in \mathbb{N}\}$.

(q) For each $n \in \mathbb{Z}$, let $V_n = \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq 1, x^n \leq y \leq \sqrt[n]{x}\}$ and $\mathscr{V} = \{V_n : n \in \mathbb{N}\}$.

☆ **2.** Which families in Exercise 1 are pairwise disjoint?

3. Prove part (b) of Theorem 2.3.1.

4. Let the universe of discourse be the set $\mathbb{R}$ of real numbers, and let $\mathscr{A}$ be the empty family of subsets of $\mathbb{R}$.

☆ **(a)** Show that $\bigcap_{A \in \mathscr{A}} A = \mathbb{R}$.

☆ **(b)** Show that $\bigcup_{A \in \mathscr{A}} A = \varnothing$.

(c) Conclude that $\bigcap_{A \in \mathscr{A}} A \subseteq \bigcup_{A \in \mathscr{A}} A$ is false in this example.

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5. ☆ **(a)** Prove parts (a) and (b) of Theorem 2.3.2.

(b) Give a direct proof of part (d) of Theorem 2.3.2 that does not use part (c).

6. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets and let B be a set. Prove that

★ **(a)** $B \cap \bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{\alpha \in \Delta} (B \cap A_\alpha)$.

(b) $B \cup \bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$.

7. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ and let $\mathcal{B} = \{B_\beta : \beta \in \Gamma\}$. Use Exercise 6 to write

★ **(a)** $\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) \cap \left(\bigcup_{\beta \in \Gamma} B_\beta\right)$ as a union of intersections.

(b) $\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \cup \left(\bigcap_{\beta \in \Gamma} B_\beta\right)$ as an intersection of unions.

8. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets, $\Delta \neq \emptyset$, and B be a set. For each part either prove the statement is true or give a counterexample.

(a) $B - \left(\bigcap_{\alpha \in \Delta} A_\alpha\right) = \bigcap_{\alpha \in \Delta} (B - A_\alpha)$.

(b) $B - \left(\bigcup_{\alpha \in \Delta} A_\alpha\right) = \bigcup_{\alpha \in \Delta} (B - A_\alpha)$.

☆ **(c)** $\left(\bigcap_{\alpha \in \Delta} A_\alpha\right) - B = \bigcap_{\alpha \in \Delta} (A_\alpha - B)$.

(d) $\left(\bigcup_{\alpha \in \Delta} A_\alpha\right) - B = \bigcup_{\alpha \in \Delta} (A_\alpha - B)$.

9. If $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ is a family of sets and $\Gamma \subseteq \Delta$, prove that

★ **(a)** $\bigcup_{\alpha \in \Gamma} A_\alpha \subseteq \bigcup_{\alpha \in \Delta} A_\alpha$.

(b) $\bigcap_{\alpha \in \Delta} A_\alpha \subseteq \bigcap_{\alpha \in \Gamma} A_\alpha$.

10. Let $\mathcal{A}$ be a nonempty family of sets.

★ **(a)** Suppose $B \subseteq A$ for every $A \in \mathcal{A}$. Prove that $B \subseteq \bigcap_{A \in \mathcal{A}} A$.

★ **(b)** What is the largest set X such that $X \subseteq A$ for all $A \in \mathcal{A}$? That is, find the set X such that (i) $X \subseteq A$ for all $A \in \mathcal{A}$; and (ii) if $V \subseteq A$ for all $A \in \mathcal{A}$, then $V \subseteq X$.

(c) Suppose $A \subseteq D$ for every $A \in \mathcal{A}$. Prove that $\bigcup_{A \in \mathcal{A}} A \subseteq D$.

(d) What is the smallest set Y such that $A \subseteq Y$ for all $A \in \mathcal{A}$? That is, find the set Y such that (i) $A \subseteq Y$ for all $A \in \mathcal{A}$; and (ii) if $A \subseteq W$ for all $A \in \mathcal{A}$, then $Y \subseteq W$.

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11. Let $X = \{1, 2, 3, 4, \ldots, 20\}$. Give an example of each of the following:

(a) a family $\mathcal{A}$ of subsets of X such that $\bigcap_{A \in \mathcal{A}} A = \{1\}$ and $\bigcup_{A \in \mathcal{A}} A = X$.

(b) a family $\mathcal{B}$ of four pairwise disjoint subsets of X such that $\bigcup_{B \in \mathcal{B}} B = X$.

(c) a family $\mathcal{C}$ of twenty pairwise disjoint subsets of X such that $\bigcup_{C \in \mathcal{C}} C = X$.

12. Give an example of an indexed collection of sets $\{A_n : n \in \mathbb{N}\}$ such that each $A_n \subseteq (0, 1)$, and for all $m, n \in \mathbb{N}$, $A_m \cap A_n \neq \emptyset$ but $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$.

13. Let $\mathcal{A}$ be a family of pairwise disjoint sets. Prove that if $\mathcal{B} \subseteq \mathcal{A}$, then $\mathcal{B}$ is a family of pairwise disjoint sets.

14. Let $\mathcal{A}$ and $\mathcal{B}$ be two pairwise disjoint families of sets. Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$.

(a) Prove that $\mathcal{C}$ is a family of pairwise disjoint sets.

(b) Give an example to show that $\mathcal{D}$ need not be pairwise disjoint.

(c) Prove that if $\bigcup_{A \in \mathcal{A}} A$ and $\bigcup_{B \in \mathcal{B}} B$ are disjoint, then $\mathcal{D}$ is pairwise disjoint.

15. Let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be a family of sets and k, m be natural numbers with $k \leq m$. Prove that

(a) $\bigcup_{i=1}^{k+1} A_i = \bigcup_{i=1}^{k} A_i \cup A_{k+1}$.

(b) $\bigcap_{i=1}^{k+1} A_i = \bigcap_{i=1}^{k} A_i \cap A_{k+1}$.

★ **(c)** $\bigcup_{i=k}^{m} A_i \subseteq \bigcup_{i=1}^{\infty} A_i$.

(d) $\bigcap_{i=1}^{\infty} A_i \subseteq \bigcap_{i=k}^{m} A_i$.

(e) $\bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{m} A_i$.

(f) $\bigcap_{i=1}^{m} A_i \subseteq \bigcap_{i=1}^{k} A_i$.

16. Suppose $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ is a family of sets such that for all $i, j \in \mathbb{N}$, if $i \leq j$, then $A_j \subseteq A_i$. (Such a family is called a **decreasing nested family** of sets.)

(a) Prove that for every $k \in \mathbb{N}$, $\bigcap_{i=1}^{k} A_i = A_k$.

(b) Prove that $\bigcup_{i=1}^{\infty} A_i = A_1$.

17. Give an example of a decreasing nested family $\{A_i : i \in \mathbb{N}\}$ (see Exercise 16) for each condition.

★ **(a)** $\bigcap_{i=1}^{\infty} A_i = [0, 1]$.

(b) $\bigcap_{i=1}^{\infty} A_i = (0, 1)$.

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(c) $\bigcap_{i=1}^{\infty} A_i = \{0, 1\}$.

(d) $\bigcap_{i=1}^{\infty} A_i = \varnothing$.

Proofs to Grade

18. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** For every indexed family $\{A_\alpha : \alpha \in \Delta\}$, $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$.

"Proof." Choose any $A_\beta \in \{A_\alpha : \alpha \in \Delta\}$. Then by Theorem 2.3.2, $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq A_\beta$ and $A_\beta \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$.

Therefore, by transitivity of set inclusion, $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$. ■

★ **(b)** **Claim.** If $A_\alpha \subseteq B$ for all $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha \subseteq B$.

"Proof." Suppose $x \in \bigcup_{\alpha\in\Delta} A_\alpha$. Then, since $A_\alpha \subseteq B$ for all $\alpha \in \Delta$, $x \in B$. Therefore, $\bigcup_{\alpha\in\Delta} A \subseteq B$. ■

(c) **Claim.** For every indexed family $\{A_\alpha : \alpha \in \Delta\}$, $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$.

"Proof." Let $\Delta = \{r, s, t\}$, $A_r = \{a, b, c, d\}$, $A_s = \{b, c, d, e\}$, $A_t = \{c, d, e, f\}$. Then $\bigcap_{\alpha\in\Delta} A_\alpha = \{c, d\} \subseteq \{a, b, c, d, e, f\} = \bigcup_{\alpha\in\Delta} A_\alpha$. ■

(d) **Claim.** For every indexed family $\{A_\alpha : \alpha \in \Delta\}$, $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$.

"Proof." Assume $\bigcap_{\alpha\in\Delta} A_\alpha \not\subseteq \bigcup_{\alpha\in\Delta} A_\alpha$. Then for some $x \in \bigcap_{\alpha\in\Delta} A_\alpha$, $x \notin \bigcup_{\alpha\in\Delta} A_\alpha$. Since $x \notin \bigcup_{\alpha\in\Delta} A_\alpha$, it is not the case that $x \in A_\alpha$ for some $\alpha \in \Delta$. Therefore, $x \notin A_\alpha$ for every $\alpha \in \Delta$. But since $x \in \bigcap_{\alpha\in\Delta} A_\alpha$, $x \in A_\alpha$ for every $\alpha \in \Delta$. This is a contradiction, so we conclude $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$. ■

★ **(e)** **Claim.** $\bigcup_{n=1}^{\infty} [n, n+1) = \mathbb{R}$.

"Proof." Let $x \in \mathbb{R}$. Choose a natural number y such that $y \le x < y + 1$. Thus $x \in [y, y + 1)$. Therefore, x is an element of $\bigcup_{n=1}^{\infty} [n, n+1)$. Since $[n, n+1) \subseteq \mathbb{R}$ for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} [n, n+1) \subseteq \mathbb{R}$.

Therefore, $\bigcup_{n=1}^{\infty} [n, n+1) = \mathbb{R}$. ■

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