Exercises 2.5

1. Use the PCI to prove that

☆ **(a)** every natural number greater than 22 can be written in the form $3s + 4t$, where s and t are integers, $s \geq 3$ and $t \geq 2$.

(b) every natural number greater than 33 can be written in the form $4s + 5t$, where s and t are integers, $s \geq 3$ and $t \geq 2$.

2. Let $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. Prove that $a_n = 2^n$ for all natural numbers n.

3. In this exercise you are to prove some well-known facts about numbers as a way of demonstrating use of the WOP. Use the WOP to prove the following:

(a) If $a > 0$, then for every natural number n, $a^n > 0$.

(b) For all positive integers a and b, $b \neq a + b$. (*Hint*: Suppose for some a there is b such that $b = a + b$. By the WOP, there is a smallest a_0 such that, for some b, $b = a_0 + b$. Apply the WOP again.)

☆ **(c)** $\sqrt{2}$ is irrational.

4. In 1202, Leonardo Fibonacci posed the following problem: Suppose a particular breed of rabbit breeds one new pair of rabbits each month, except that a 1-month-old pair is too young to breed. Suppose further that no rabbit breeds with any other except its paired mate and that rabbits live forever. At 1 month we have our original pair of rabbits. At 2 months we still have the single pair. At 3 months, we have two pairs (the original and their one pair of offspring). At 4 months we have three pairs (the original pair, one older pair of offspring, and one new pair of offspring).

(a) Show that at n months, there are f_n pairs of rabbits.

(b) Calculate the first ten Fibonacci numbers $f_1, f_2, f_3, \ldots, f_{10}$.

(c) Find a formula for $f_{n+3} - f_{n+1}$.

5. Use the PMI to show that each of the following statements about Fibonacci numbers is true:

☆ **(a)** f_{3n} is even and both f_{3n+1} and f_{3n+2} are odd for all natural numbers n.

(b) $\gcd(f_n, f_{n+1}) = 1$ for all natural numbers n.

(c) $\gcd(f_n, f_{n+2}) = 1$ for all natural numbers n.

★ **(d)** $f_1 + f_2 + f_3 + \cdots + f_n = f_{n+2} - 1$ for all natural numbers n.

6. Use the PCI to prove the following properties of Fibonacci numbers:

(a) f_n is a natural number for all natural numbers n.

(b) $f_{n+6} = 4f_{n+3} + f_n$ for all natural numbers n.

(c) For any natural number a, $f_a f_n + f_{a+1} f_{n+1} = f_{a+n+1}$ for all natural numbers n.

☆ **(d)** (Binet's formula) Let α be the positive solution and β the negative solution to the equation $x^2 = x + 1$. (The values are $\alpha = \dfrac{1 + \sqrt{5}}{2}$ and $\beta = \dfrac{1 - \sqrt{5}}{2}$.) Show for all natural numbers n that

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

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☆ **7.** Complete the proof of the Division Algorithm (Theorem 2.5.1) for the case $a < 0$. That is, show that for all integers a and b, with $a < 0$, there exist unique integers q and r such that $b = aq + r$ and $0 \leq r < |a| = -a$.

8. Let a and b be nonzero integers. Prove that there is a smallest positive linear combination of a and b. (Theorem 2.5.2)

☆ **9.** Prove the Fundamental Theorem of Arithmetic: Every natural number greater than 1 is prime or can be expressed uniquely as a product of primes.

10. In the tournament described in Exercise 12 of Section 2.4, a top player is defined to be one who either beats every other player or beats someone who beats the other player. Use the WOP to show that in every such tournament with n players ($n \in \mathbb{N}$), there is at least one top player.

11. Let the "Fibonacci-2" numbers g_n be defined as follows:

$$g_1 = 2,\ g_2 = 2, \text{ and } g_{n+2} = g_{n+1}g_n \quad \text{for all } n \geq 1.$$

(a) Calculate the first five "Fibonacci-2" numbers.
(b) Show that $g_n = 2^{f_n}$.

12. Complete the proof of the equivalences of the PMI, PCI, and WOP by
(a) using the PCI to prove the WOP.
★ **(b)** using the WOP to prove the PMI.
(c) using the WOP to prove the PCI.

Proofs to Grade

13. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) **Claim.** For all natural numbers n, 5 divides $8^n - 3^n$.
"Proof." Suppose there is $n \in \mathbb{N}$ such that 5 does not divide $8^n - 3^n$. Then by the WOP there is a smallest such natural number t. Now $t \neq 1$ since 5 does divide $8^1 - 3^1$. Therefore $t - 1$ is a natural number smaller than t, so 5 divides $8^{t-1} - 3^{t-1}$. But then 5 divides $8(8^{t-1} - 3^{t-1})$ and 5 divides $5(3^{t-1})$, so 5 divides their sum, which is $8^t - 3^t$. This is a contradiction. Therefore, 5 divides $8^n - 3^n$ for all $n \in \mathbb{N}$. ■

★ **(b)** **Claim.** For every natural number n, 3 divides $n^3 + 2n + 1$.
"Proof." Suppose there is a natural number n such that 3 does not divide $n^3 + 2n + 1$. By the WOP, there is a smallest such number. Call this number m. Then $m - 1$ is a natural number and 3 does divide

$$\begin{aligned}(m-1)^3 + 2(m-1) + 1 &= m^3 - 3m^2 + 3m - 1 + 2m - 2 + 1 \\ &= m^3 - 3m^2 + 5m - 2.\end{aligned}$$

But 3 also divides $3m^2 - 3m + 3$, so 3 divides the sum of these two expressions, which is $m^3 + 2m + 1$. This contradicts what we know about m. Therefore, the set $\{n \in \mathbb{N}$: 3 does not divide $n^3 + 2n + 1\}$ must be empty. Therefore, 3 divides $n^3 + 2n + 1$, for every natural number n. ■

(c) **Claim.** The PCI implies the WOP.
"Proof." Assume the PCI. Let T be a nonempty subset of $\mathbb{N}$. Then T has some element x. Then $\{1, 2, \ldots, x - 1\} \subseteq \mathbb{N} - T$. By the PCI,

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$x \in \mathbb{N} - T$. This is a contradiction, because $x \in T$ and $x \in \mathbb{N} - T$. Therefore T has a smallest element. ■

(d) **Claim.** The WOP implies the PCI.

"Proof." Assume the WOP. To prove the PCI, let S be a subset of $\mathbb{N}$ such that for all natural numbers m, $\{1, 2, 3, \ldots, m-1\} \subseteq S$. Let $k \in \mathbb{N}$. Then $k+1$ is an integer, so $\{1, 2, \ldots, (k+1)-1\} \subseteq S$. However, $(k+1)-1 = k$, so $k \in S$. Thus every natural number is in S, so $S = \mathbb{N}$. ■

2.6 Principles of Counting

Recall from the *Preface to the Student* that a set A is **finite** iff $A = \varnothing$ or A has n elements for some $n \in \mathbb{N}$. For a finite set A the number of elements in A is denoted $\overline{\overline{A}}$. For example, if $A = \{p, q\}$ and $B = \{3, 2, 1, 5, 9\}$ then $\overline{\overline{A}} = 2$ and $\overline{\overline{B}} = 5$. This section describes some of the fundamental techniques for counting the number of elements in finite sets.

A more precise development of the concepts of the cardinality (number of elements) and finiteness of a set appears in Chapter 5. For this reason, proofs of the basic counting rules in this section appear in Section 5.1.

Theorem 2.6.1 The Sum Rule

Let A and B be finite sets with m and n elements, respectively. If A and B are disjoint then $\overline{\overline{A \cup B}} = m + n$.

We use the Sum Rule so often, we don't have to think about it: If a basket has 5 oranges and 6 apples, then there are 11 pieces of fruit in the basket. The rule can be extended to any finite number of sets. We prove the Generalized Sum Rule by using the Principle of Mathematical Induction.

Theorem 2.6.2 The Generalized Sum Rule

For all $n \in \mathbb{N}$ and for every family $\mathcal{A} = \{A_i : i = 1, 2, 3, \ldots, n\}$ of n distinct pairwise disjoint sets, if $\overline{\overline{A_i}} = a_i$ for $1 < i < n$, then

$$\overline{\overline{\bigcup_{i=1}^{n} A_i}} = \sum_{i=1}^{n} a_i.$$

Proof. The proof is by induction on the number n of sets in the family $\mathcal{A}$.

(i) If $n = 1$, then $\overline{\overline{\bigcup_{i=1}^{1} A_i}} = \overline{\overline{A_1}} = a_1 = \sum_{i=1}^{1} a_i$.

(ii) Suppose for some $n \in \mathbb{N}$ that $\overline{\overline{A_i}} = a_i$ for $i = 1, 2, \ldots, n$ and $\overline{\overline{\bigcup_{i=1}^{n} A_i}} = \sum_{i=1}^{n} a_i$ for every family $\mathcal{A} = \{A_i : i = 1, 2, 3, \ldots, n\}$ of n distinct pairwise disjoint sets. Let $\mathcal{A} = \{A_1, A_2, A_3, \ldots, A_{n+1}\}$ be a family of $n+1$ distinct pairwise disjoint sets with $\overline{\overline{A_i}} = a_i$ for $i = 1, 2, \ldots, n+1$. Then

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