The **join** of these two tables, denoted $R \otimes S$, is a table with 7 columns. The rows of the table are obtained by merging 5-tuples from R and 3-tuples from S that share a common ID number:

$R \otimes S$

First Name	Last Name	Student ID	Residence Hall	Room Number	Tuition	Room Charges
Krista	Maire	1234	Orlando	77	$80	$40
Harold	Dorman	2490	Mountain	455	$150	$20
Ferlin	Husky	5555	Dove	213A	$75	$25
Martha	Reeves	3215	Vandella	238	$0	$0
Kim	Anen	6920	Bowie	1979	$0	$60

The join operation is one of several database operations that allow a manager to create tables in response to requests for information (queries). There are many advantages to storing data in simple tables like R and S, but requests such as "What is the room charge for Harold Dorman?" cannot be answered using either of the tables by itself.

Exercises 3.1

1. Let T be the relation $\{(3, 1), (2, 3), (3, 5), (2, 2), (1, 6), (2, 6), (1, 2)\}$. Find
(a) Dom (T). **(b)** Rng (T).
(c) T^{-1}. **(d)** $(T^{-1})^{-1}$.

2. Find the domain and range for the relation W on $\mathbb{R}$ given by $x\ W\ y$ iff
★ **(a)** $y = 2x + 1$. **(b)** $y = x^2 + 3$.
★ **(c)** $y = \sqrt{x - 1}$. **(d)** $y = \dfrac{1}{x^2}$
★ **(e)** $y \leq x^2$. **(f)** $|x| < 2$ and $y = 3$.
(g) $|x| < 2$ or $y = 3$. **(h)** $y \neq x$.

☆ **3.** Sketch the graph of each relation in Exercise 2.

4. The inverse of $R = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = 2x + 1\}$ may be expressed in the form $R^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{x - 1}{2}\right\}$, the set of all pairs (x, y) subject to some condition. Use this form to give the inverses of the following relations. In (i), (j), and (k), P is the set of all people.
★ **(a)** $R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x\}$
(b) $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = -5x + 2\}$
★ **(c)** $R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = 7x - 10\}$
(d) $R_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2 + 2\}$
★ **(e)** $R_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = -4x^2 + 5\}$

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(f) $R_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y < x + 1\}$

★ **(g)** $R_7 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y > 3x - 4\}$

(h) $R_8 = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{2x}{x - 2}\right\}$

★ **(i)** $R_9 = \{(x, y) \in P \times P: y \text{ is the father of } x\}$

(j) $R_{10} = \{(x, y) \in P \times P: y \text{ is a sibling of } x\}$

(k) $R_{11} = \{(x, y) \in P \times P: y \text{ loves } x\}$

5. Let $R = \{(1, 5), (2, 2), (3, 4), (5, 2)\}$, $S = \{(2, 4), (3, 4), (3, 1), (5, 5)\}$, and $T = \{(1, 4), (3, 5), (4, 1)\}$. Find

(a) $R \circ S$.
★ **(b)** $R \circ T$.
(c) $T \circ S$.
★ **(d)** $R \circ R$.
(e) $S \circ R$.
(f) $T \circ T$.
(g) $R \circ (S \circ T)$.
(h) $(R \circ S) \circ T$.

6. Find these composites for the relations defined in Exercise 4.

★ **(a)** $R_1 \circ R_1$
(b) $R_1 \circ R_2$
(c) $R_2 \circ R_2$
★ **(d)** $R_2 \circ R_3$
(e) $R_2 \circ R_4$
(f) $R_4 \circ R_2$
★ **(g)** $R_4 \circ R_5$
(h) $R_6 \circ R_2$
(i) $R_6 \circ R_4$
★ **(j)** $R_6 \circ R_6$
(k) $R_7 \circ R_7$
(l) $R_5 \circ R_5$
(m) $R_8 \circ R_8$
★ **(n)** $R_3 \circ R_8$
(o) $R_8 \circ R_3$
☆ **(p)** $R_9 \circ R_9$

7. Give the digraphs for these relations on the set $\{1, 2, 3\}$.

(a) $=$
(b) $S = \{(1, 3), (2, 1)\}$
(c) $\leq$
(d) S^{-1}, where $S = \{(1, 3), (2, 1)\}$
(e) $\neq$
(f) $S \circ S$, where $S = \{(1, 3), (2, 1)\}$

8. Let $A = \{a, b, c, d\}$. Give an example of relations R, S, and T on A such that

(a) $R \circ S \neq S \circ R$.
(b) $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$.
(c) $S \circ R = T \circ R$ but $S \neq T$.
(d) R and S are nonempty, and $R \circ S$ and $S \circ R$ are empty.

9. Let R be a relation from A to B and S be a relation from B to C.

(a) Prove that $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$.
(b) Show by example that $\text{Dom}(S \circ R) = \text{Dom}(R)$ may be false.
(c) Which of these two statements must be true:

$$\text{Rng}(S) \subseteq \text{Rng}(S \circ R) \quad \text{or} \quad \text{Rng}(S \circ R) \subseteq \text{Rng}(S)?$$

Give an example to show that the other statement may be false.

10. Complete the proof of Theorem 3.1.3.

11. Show by example that $(A \times B) \times C = A \times (B \times C)$ may be false.

12. Prove that if A has m elements and B has n elements, then there are 2^{mn} different relations from A to B.

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13. **(a)** Let R be a relation from A to B. For $a \in A$, define the **vertical section of R at a** to be $R_a = \{y \in B: (a, y) \in R\}$. Prove that $\bigcup_{a \in A} R_a = \text{Rng } (R)$.

(b) Let R be a relation from A to B. For $a \in A$, define the **horizontal section of R at b** to be ${}_bR = \{x \in A: (x, b) \in R\}$. Prove that $\bigcup_{b \in B} {}_bR = \text{Dom } (R)$.

14. We may define ordered triples in terms of ordered pairs by saying that $(a, b, c) = ((a, b), c)$. Use this definition to prove that $(a, b, c) = (x, y, z)$ iff $a = x$ and $b = y$ and $c = z$.

Proofs to Grade

15. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** $(A \times B) \cup C = (A \times C) \cup (B \times C)$.

"Proof." $x \in (A \times B) \cup C$
iff $x \in A \times B$ or $x \in C$
iff $x \in A$ and $x \in B$ or $x \in C$
iff $x \in A \times C$ or $x \in B \times C$
iff $x \in (A \times C) \cup (B \times C)$. ■

★ **(b)** **Claim.** If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.

"Proof." Suppose $A \times C \not\subseteq B \times D$. Then there exists $(a, c) \in A \times C$ with $(a, c) \notin B \times D$. But $(a, c) \in A \times C$ implies that $a \in A$ and $c \in C$, whereas $(a, c) \notin B \times D$ implies that $a \notin B$ and $c \notin D$. However, $A \subseteq B$ and $C \subseteq D$, so $a \in B$ and $c \in D$. This is a contradiction. Therefore, $A \times C \subseteq B \times D$. ■

(c) **Claim.** If $A \times B = A \times C$ and $A \neq \emptyset$, then $B = C$.

"Proof." Suppose $A \times B = A \times C$. Then

$$\frac{A \times B}{A} = \frac{A \times C}{A}.$$

Therefore $B = C$. ■

★ **(d)** **Claim.** If $A \times B = A \times C$ and $A \neq \emptyset$, then $B = C$.

"Proof." To show $B = C$, suppose $b \in B$. Choose any $a \in A$. Then $(a, b) \in A \times B$. But since $A \times B = A \times C$, $(a, b) \in A \times C$. Thus $b \in C$. This proves $B \subseteq C$. A proof of $C \subseteq B$ is similar. Therefore, $B = C$. ■

(e) **Claim.** Let R and S be relations from A to B and from B to C, respectively. Then $S \circ R = (R \circ S)^{-1}$.

"Proof." The pair $(x, y) \in S \circ R$ iff $(y, x) \in R \circ S$ iff $(x, y) \in (R \circ S)^{-1}$ Therefore, $S \circ R = (R \circ S)^{-1}$. ■

(f) **Claim.** Let R be a relation from A to B. Then $I_A \subseteq R^{-1} \circ R$.

"Proof." Suppose $(x, x) \in I_A$. Choose any $y \in B$ such that $(x, y) \in R$. Then, $(y, x) \in R^{-1}$. Thus $(x, x) \in R^{-1} \circ R$. Therefore, $I_A \subseteq R^{-1} \circ R$. ■

(g) **Claim.** Suppose R is a relation from A to B. Then $R^{-1} \circ R \subseteq I_A$.

"Proof." Let $(x, y) \in R^{-1} \circ R$. Then for some $z \in B$, $(x, z) \in R$ and $(z, y) \in R^{-1}$. Thus $(y, z) \in R$. Since $(x, z) \in R$ and $(y, z) \in R$, $x = y$. Thus $(x, y) = (x, x)$ and $x \in A$, so $(x, y) \in I_A$. ■

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