modulo 12 that we routinely do arithmetic with them: 9 hours after 8 o'clock is 5 o'clock, because $8 + 9 = 17$ and $17 = 5$ (modulo 12) and 4 hours before 3 o'clock is 11 o'clock, because $3 - 4 = -1 = 11$ (modulo 12).

Our next theorem will show that there are always m different equivalence classes for the relation $\equiv_m$ and the set $\mathbb{Z}_m$ is always $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$. It is helpful to observe that $0, 1, 2, \ldots$, and $m - 1$ are exactly all the possible remainders when integers are divided by m. For this reason the elements of $\mathbb{Z}_m$ are sometimes called the *residue* (or remainder) classes modulo m.

Theorem 3.2.3 Let m be a fixed positive integer. Then

(a) For integers x and y, $x = y \pmod m$ iff the remainder when x is divided by m equals the remainder when y is divided by m.

(b) $\mathbb{Z}_m$ consists of m distinct equivalence classes: $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$.

Proof.

(a) Let x and y be integers. By the Division Algorithm, there exist integers q, r, t, and s such that $x = mq + r$, with $0 \leq r < m$ and $y = mt + s$, with $0 \leq s < m$. ⟨*We must show that* $x = y \pmod m$ *iff* $r = s$.⟩ Then

$$\begin{aligned} x = y \pmod m \text{ iff } & m \text{ divides } x - y \\ \text{iff } & m \text{ divides } (mq + r) - (mt + s) \\ \text{iff } & m \text{ divides } m(q - t) + (r - s) \\ \text{iff } & m \text{ divides } r - s \\ \text{iff } & r = s. \ \langle \textit{This is because } 0 \leq r < m \textit{ and } 0 \leq s < m. \rangle \end{aligned}$$

(b) ⟨*We first show that* $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$.⟩ For each k, where $0 \leq k \leq m - 1$, the set $\overline{k}$ is an equivalence class, so $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$ is a subset of $\mathbb{Z}_m$. Now suppose $\overline{x} \in \mathbb{Z}_m$ for some integer x. By the Division Algorithm, there exist integers q and r such that $x = mq + r$, with $0 \leq r < m$. Then $x - r = mq$, so m divides $x - r$. Thus $x = r \pmod m$. By Theorem 3.2.1(b) $\overline{x} = \overline{r}$. Therefore $\mathbb{Z}_m \subseteq \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$.

Finally we will know that $\mathbb{Z}_m$ has exactly m elements when we show that the equivalence classes $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}$ are all distinct. Suppose $\overline{k} = \overline{r}$ where $0 \leq r \leq k \leq m - 1$. Then $k = r \pmod m$, and thus m divides $k - r$. But $0 \leq k - r \leq m - 1$, so $k - r = 0$. Then $k = r$. Therefore the m equivalence classes are distinct. ■

Exercises 3.2

1. Indicate which of the following relations on the given sets are reflexive on a given set, which are symmetric, and which are transitive.

★ **(a)** $\{(1, 2)\}$ on $\{1, 2\}$ **(b)** $\leq$ on $\mathbb{N}$

(c) $=$ on $\mathbb{N}$ **(d)** $<$ on $\mathbb{N}$

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★ **(e)** $\geq$ on $\mathbb{N}$ **(f)** $\neq$ on $\mathbb{N}$
(g) "divides" on $\mathbb{N}$ **(h)** $\{(x, y) \in \mathbb{Z} \times \mathbb{Z}: x + y = 10\}$
(i) $\{(1, 5), (5, 1), (1, 1)\}$ on the set $A = \{1, 2, 3, 4, 5\}$
(j) $\perp = \{(l, m)$: l and m are lines and l is perpendicular to $m\}$ on the set of all lines in a plane
(k) R, where $(x, y)\ R\ (z, w)$ iff $x + z \leq y + w$, on the set $\mathbb{R} \times \mathbb{R}$
★ **(l)** S, where $x\ S\ y$ iff x is a sibling of y, on the set P of all people
(m) T, where $(x, y)\ T\ (z, w)$ iff $x + y \leq z + w$, on the set $\mathbb{R} \times \mathbb{R}$

2. Let $A = \{1, 2, 3\}$. List the ordered pairs and draw the digraph of a relation on A with the given properties.
★ **(a)** not reflexive, not symmetric, and not transitive
(b) reflexive, not symmetric, and not transitive
(c) not reflexive, symmetric, and not transitive
★ **(d)** reflexive, symmetric, and not transitive
(e) not reflexive, not symmetric, and transitive
(f) reflexive, not symmetric, and transitive
(g) not reflexive, symmetric, and transitive
(h) reflexive, symmetric, and transitive

☆ **3.** For each part of Exercise 2, give an example of a relation on $\mathbb{R}$ with the desired properties.

4. Let R be a relation on a set A. Prove that
(a) if A is nonempty, the empty relation $\varnothing$ is not reflexive on A.
(b) the empty relation $\varnothing$ is symmetric and transitive for every set A.

5. For each of the following, prove that the relation is an equivalence relation. Then give information about the equivalence classes as specified.
(a) The relation R on $\mathbb{R}$ given by $x\ R\ y$ iff $x - y \in \mathbb{Q}$. Give the equivalence class of 0; of 1, of $\sqrt{2}$.
(b) The relation R on $\mathbb{N}$ given by $m\ R\ n$ iff m and n have the same digit in the tens places. Find an element of $106/R$ that is less than 50; between 150 and 300; greater than 1,000. Find three such elements in the equivalence class $635/R$.
(c) The relation V on $\mathbb{R}$ given by $x\ V\ y$ iff $x = y$ or $xy = 1$. Give the equivalence class of 3; of $-\frac{2}{3}$; of 0.
☆ **(d)** On $\mathbb{N}$, the relation R given by $a\ R\ b$ iff the prime factorizations of a and b have the same number of 2's. For example, $16\ R\ 80$ because $16 = 2^4$ and $80 = 2^4 \cdot 5$. Name three elements in each of these classes: $1/R$, $4/R$, $72/R$.
(e) The relation T on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)\ T\ (a, b)$ iff $x^2 + y^2 = a^2 + b^2$. Sketch the equivalence class of (1, 2); of (4, 0).
(f) For the set $X = \{m, n, p, q, r, s\}$, let R be the relation on $\mathcal{P}(X)$ given by $A\ R\ B$ iff A and B have the same number of elements. List all the elements in $\{m\}/R$; in $\{m, n, p, q, r, s\}/R$. How many elements are in X/R? How many elements are in $\mathcal{P}(X)/R$?

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☆ **(g)** The relation P on $\mathbb{R} \times \mathbb{R}$ defined by $(x, y)\, P\, (z, w)$ iff $|x - y| = |z - w|$. Name at least one ordered pair in each quadrant that is related to (3, 0). Describe all ordered pairs in the equivalence class of (0, 0); in the class of (1, 0).

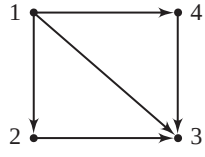
(h) Let R be the relation on the set of all differentiable functions defined by $f\, R\, g$ iff f and g have the same first derivative, that is, $f' = g'$. Name three elements in each of these classes: x^2/R, $(4x^3 + 10x)/R$. Describe x^3/R and $7/R$.

(i) The relation T on $\mathbb{R}$ given by $x\, T\, y$ iff $\sin x = \sin y$. Describe the equivalence class of 0; of $\pi/2$; of $\pi/4$.

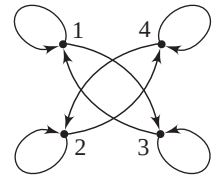
6. Let R be the relation on $\mathbb{Q}$ defined by $\frac{p}{q}\, R\, \frac{s}{t}$ iff $pt = qs$. Show that R is an equivalence relation. Describe all ordered pairs in the equivalence class of $\frac{2}{3}$.

7. Which of these digraphs represent relations that are (i) reflexive? (ii) symmetric? (iii) transitive?

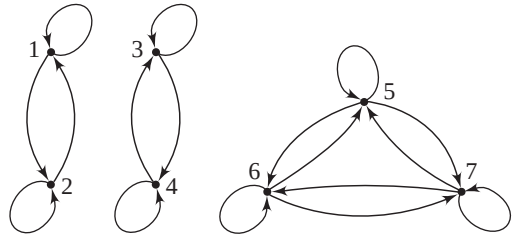
★ **(a)**



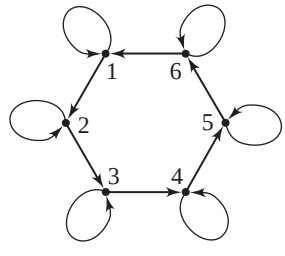
(b)



★ **(c)**



(d)



8. Determine the equivalence classes for the relation of

★ **(a)** congruence modulo 5. **(b)** congruence modulo 8.

(c) congruence modulo 1. **(d)** congruence modulo 7.

9. Name a positive integer and a negative integer that are

(a) congruent to 0 (mod 5) and not congruent to 0 (mod 6).

(b) congruent to 0 (mod 5) and congruent to 0 (mod 6).

(c) congruent to 2 (mod 4) and congruent to 8 (mod 6).

(d) congruent to 3 (mod 4) and congruent to 3 (mod 5).

(e) congruent to 1 (mod 3) and congruent to 1 (mod 7).

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10. Using the fact that $\equiv_m$ is an equivalence relation on $\mathbb{Z}$ and without reference to Theorems 3.2.1 and 3.2.3, prove that for all x and y in $\mathbb{Z}$:

(a) $x \in \bar{x}$. ☆ **(b)** $\bar{x} \neq \emptyset$.

☆ **(c)** if $x \equiv_m y$, then $\bar{x} = \bar{y}$. **(d)** if $\bar{x} = \bar{y}$, then $x \equiv_m y$.

(e) if $\bar{x} \cap \bar{y} \neq \emptyset$, then $\bar{x} = \bar{y}$. **(f)** if $\bar{x} \cap \bar{y} = \emptyset$, then $\bar{x} \neq \bar{y}$.

11. Consider the relation S on $\mathbb{N}$ defined by $x\ S\ y$ iff 3 divides $x + y$. Prove that S is not an equivalence relation.

12. Suppose that R and S are equivalence relations on a set A. Prove that $R \cap S$ is an equivalence relation on A.

13. The properties of reflexivity, symmetry, and transitivity are related to the identity relation and the operations of inversion and composition. Prove that

(a) R is a reflexive relation on A iff $I_A \subseteq R$.

★ **(b)** R is symmetric iff $R = R^{-1}$.

(c) R is transitive iff $R \circ R \subseteq R$.

14. Prove that if R is a symmetric, transitive relation on A and the domain of R is A, then R is reflexive on A.

15. Let R be a relation on the set A.

★ **(a)** Prove that $R \cup R^{-1}$ is symmetric. ($R \cup R^{-1}$ is the **symmetric closure** of R.)

(b) Prove that if S is a symmetric relation on A and $R \subseteq S$, then $R^{-1} \subseteq S$.

16. Let R be a relation on the set A. Define $T_R = \{(x, y) \in A \times A$: for some $n \in \mathbb{N}$ there exists $a_0 = x, a_1, a_2, \ldots, a_n = y \in A$ such that $(a_0, a_1), (a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n) \in R\}$.

(a) Prove that T_R is transitive. (T_R is the **transitive closure** of R.)

(b) Prove that if S is a transitive relation on A and $R \subseteq S$, then $T_R \subseteq S$.

17. The **complement** of a digraph has the same vertex set as the original digraph, and an arc from x to y exactly when the original digraph does not have an arc from x to y. The two digraphs shown below are complementary. Call a digraph symmetric (transitive) iff its relation is symmetric (transitive).



(a) Show that the complement of a symmetric digraph is symmetric.

(b) Show by example that the complement of a transitive digraph need not be transitive.

☆ **18.** Let L be a relation on a set A that is reflexive on A and transitive but not necessarily symmetric. Let R be the relation defined on A by $x\ R\ y$ iff $x\ L\ y$ and $y\ L\ x$. Prove that R is an equivalence relation.

Proofs to Grade

19. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

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(a) **Claim.** If the relation R is symmetric and transitive, it is also reflexive.
"Proof." Since R is symmetric, if $(x, y) \in R$, then $(y, x) \in R$. Thus $(x, y) \in R$ and $(y, x) \in R$, and since R is transitive, $(x, x) \in R$. Therefore, R is reflexive. ■

(b) **Claim.** The relation T on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)\ T\ (r, s)$ iff $x + y = r + s$ is symmetric.
"Proof." Suppose $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then $(x, y)\ T\ (y, x)$ because $x + y = y + x$. Therefore, T is symmetric. ■

(c) **Claim.** The relation W on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)\ W\ (r, s)$ iff $x - r = y - s$ is symmetric.
"Proof." Suppose (x, y) and (r, s) are in $\mathbb{R} \times \mathbb{R}$ and $(x, y)\ W\ (r, s)$. Then $x - r = y - s$. Therefore, $r - x = s - y$, so $(r, s)\ W\ (x, y)$. Thus W is symmetric. ■

(d) **Claim.** If the relations R and S are symmetric, then $R \cap S$ is symmetric.
"Proof." Let R be the relation of congruence modulo 10 and S the relation of congruence modulo 6 on the integers. Both R and S are symmetric. If $(x, y) \in R \cap S$, then 6 and 10 divide $x - y$. Therefore, 2, 3, and 5 all divide $x - y$, so 30 divides $x - y$. Also if 30 divides $x - y$, then 6 and 10 divide $x - y$, so $R \cap S$ is the relation of congruence modulo 30. Therefore, $R \cap S$ is symmetric. ■

(e) **Claim.** If the relations R and S are symmetric, then $R \cap S$ is symmetric.
"Proof." Suppose $(x, y) \in R \cap S$. Then $(x, y) \in R$ and $(x, y) \in S$. Since R and S are symmetric, $(y, x) \in R$ and $(y, x) \in S$. Therefore, $(y, x) \in R \cap S$. ■

★ **(f)** **Claim.** If the relations R and S are transitive, then $R \cap S$ is transitive.
"Proof." Suppose $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. Then $(x, y) \in R$ and $(y, z) \in S$. Therefore, $(x, z) \in R \cap S$. ■

3.3 Partitions

Partitioning is frequently used to organize the world around us. The United States, for example, is partitioned in several ways—by postal zip codes, state boundaries, time zones, etc. In each case nonempty subsets of the United States are defined that do not overlap and that together comprise the entire country. This section introduces this concept of partitioning of a set and describes the close relationship between partitions and equivalence relations.

DEFINITION Let A be a nonempty set. $\mathcal{P}$ is a **partition of A** iff $\mathcal{P}$ is a set of subsets of A such that

(i) If $X \in \mathcal{P}$, then $X \neq \emptyset$.

(ii) If $X \in \mathcal{P}$ and $Y \in \mathcal{P}$, then $X = Y$ or $X \cap Y = \emptyset$.

(iii) $\bigcup_{X \in \mathcal{P}} X = A$.

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