To show $\mathcal{P} \subseteq A/Q$, let $B \in \mathcal{P}$. As an element of a partition, $B \neq \emptyset$. Choose any $t \in B$; then we claim $B = t/Q$. If $s \in B$, then $t\ Q\ s$, so $s \in t/Q$. On the other hand, if $s \in t/Q$, then $t\ Q\ s$; so s and t are elements of the same member of $\mathcal{P}$, which must be B. ■

Example. Let $A = \{1, 2, 3, 4\}$ and $\mathcal{P} = \{\{1\}, \{2, 3\}, \{4\}\}$ be a partition of A with three sets. The equivalence relation Q associated with $\mathcal{P}$ is $\{(1, 1), (2, 2), (3, 3), (4, 4), (2. 3), (3, 2)\}$. The three equivalence classes for Q are $1/Q = \{1\}$, $2/Q = 3/Q = \{2, 3\}$, and $4/Q = \{4\}$. The set of all equivalence classes is precisely $\mathcal{P}$.

Example. The set $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$ is a partition of $\mathbb{Z}$, where

$$A_0 = \{4k : k \in \mathbb{Z}\}.$$
$$A_1 = \{4k + 1 : k \in \mathbb{Z}\}.$$
$$A_2 = \{4k + 2 : k \in \mathbb{Z}\}.$$
$$A_3 = \{4k + 3 : k \in \mathbb{Z}\}.$$

Then integers x and y are in the same set A_i iff $x = 4n + i$ and $y = 4m + i$ for some integers n and m or, in other words, iff $x - y$ is a multiple of 4. Thus, the equivalence relation associated with the partition $\mathcal{A}$ is the relation of congruence modulo 4 and each A_i is the residue class of i modulo 4, for $i = 0, 1, 2, 3$.

We have seen that every equivalence relation on a set determines a partition for the set and every partition of a set determines a corresponding equivalence relation on that set. Furthermore, if we start with an equivalence relation, the partition we make is the set of equivalence classes, and if we use that partition to form an equivalence relation, the relation formed is the relation we started with. Thus, each concept may be used to describe the other. This is to our advantage, for we may use partitions and equivalence relations interchangeably, choosing the one that lends itself more readily to the situation at hand.

Exercises 3.3

1. Describe four different partitions of the set of all students enrolled at a university.

2. For the given set A, determine whether $\mathcal{P}$ is a partition of A.

(a) $A = \{1, 2, 3, 4\}$, $\mathcal{P} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$

(b) $A = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{P} = \{\{1, 2\}, \{3\}, \{4, 5\}\}$

(c) $A = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{P} = \{\{1, 3\}, \{5, 6\}, \{2, 4\}, \{7\}\}$

★ **(d)** $A = \mathbb{N}$, $\mathcal{P} = \{1, 2, 3, 4, 5\} \cup \{n \in \mathbb{N} : n > 5\}$

(e) $A = \mathbb{R}$, $\mathcal{P} = (-\infty, -1) \cup [-1, 1] \cup (1, \infty)$

(f) $A = \mathbb{R}$, $\mathcal{P} = \{S_y : y \in \mathbb{R} \text{ and } y > 0\}$, where $S_y = \{x \in \mathbb{R} : x < y\}$

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3. Describe the partition for each of the following equivalence relations.
- **(a)** For $x, y \in \mathbb{R}$, $x\,R\,y$ iff $x - y \in \mathbb{Z}$.
- ★ **(b)** For $n, m \in \mathbb{Z}$, $n\,R\,m$ iff n and m have the same tens digit.
- **(c)** For $x, y \in \mathbb{R}$, $x\,R\,y$ iff $\sin x = \sin y$.
- **(d)** For $x, y \in \mathbb{R}$, $x\,R\,y$ iff $x^2 = y^2$.
- **(e)** For (x, y) and $(u, v) \in \mathbb{R} \times \mathbb{R}$, $(x, y)\,S\,(u, v)$ iff $xy = uv = 0$ or $xyuv > 0$.
- **(f)** $(x, y)\,R\,(u, v)$ iff $x + v = y + u$.

4. Let $C = \{i, -1, -i, 1\}$, where $i^2 = -1$. The relation R on C given by $x\,R\,y$ iff $xy = \pm 1$ is an equivalence relation on C. Give the partition of C associated with R.

☆ **5.** Let C be as in Exercise 4. The relation S on $C \times C$ given by $(x, y)\,S\,(u, v)$ iff $xy = uv$ is an equivalence relation. Give the partition of $C \times C$ associated with S.

6. Describe the equivalence relation on each of the following sets with the given partition.
- **(a)** $\mathbb{N}$, $\{\{1, 2, \ldots 9\}, \{10, 11, \ldots 99\}, \{100, 101, \ldots 999\}, \ldots\}$
- ★ **(b)** $\mathbb{Z}$, $\{\ldots, \{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}, \{3, 4, 5, \ldots\}\}$
- **(c)** $\mathbb{R}$, $\{(-\infty, 0), \{0\}, (0, \infty)\}$
- ★ **(d)** $\mathbb{R}$, $\{\ldots, (-3, -2), \{-2\}, (-2, -1), \{-1\}, (-1, 0), \{0\}, (0, 1), \{1\}, (1, 2), \{2\}, (2, 3), \ldots\}$
- **(e)** $\mathbb{Z}$, $\{A, B\}$, where $A = \{x \in \mathbb{Z}: x < 3\}$ and $B = \mathbb{Z} - A$

7. For each $a \in \mathbb{R}$, let $A_a = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = a - x^2\}$.
- **(a)** Sketch a graph of the set A_a for $a = -2, -1, 0, 1$, and 2.
- **(b)** Prove that $\{A_a: a \in \mathbb{R}\}$ is a partition of $\mathbb{R} \times \mathbb{R}$.
- **(c)** Describe the equivalence relation associated with this partition.

8. List the ordered pairs in the equivalence relation on $A = \{1, 2, 3, 4, 5\}$ associated with these partitions:
- ★ **(a)** $\{\{1, 2\},\{3, 4, 5\}\}$
- **(b)** $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$
- **(c)** $\{\{2, 3, 4, 5\}, \{1\}\}$

9. Partition the set $D = \{1, 2, 3, 4, 5, 6, 7\}$ into two subsets: those symbols made from straight line segments only (like 4), and those that are drawn with at least one curved segment (like 2). Describe or draw the digraph of the corresponding equivalence relation on D.

10. Complete the proof of Theorem 3.3.2 by proving that if $\mathcal{P}$ is a partition of A, and $x\,Q\,y$ iff there exists $C \in \mathcal{P}$ such that $x \in C$ and $y \in C$, then
- **(a)** Q is symmetric.
- **(b)** Q is reflexive on A.

★ **11.** Let R be a relation on a set A that is reflexive and symmetric but not transitive. Let $R(x) = \{y: x\,R\,y\}$. [Note that $R(x)$ is the same as x/R except that R is not an equivalence relation in this exercise.] Does the set $\mathcal{A} = \{R(x): x \in A\}$ always form a partition of A? Prove that your answer is correct.

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12. Repeat Exercise 11, assuming R is reflexive and transitive but not symmetric.

13. Repeat Exercise 11, assuming R is symmetric and transitive but not reflexive.

14. Let A be a set with at least three elements.

★ **(a)** If $\mathcal{P} = \{B_1, B_2\}$ is a partition of A with $B_1 \neq B_2$, is $\{B_1^c, B_2^c\}$ a partition of A? Explain. What if $B_1 = B_2$?

(b) If $\mathcal{P} = \{B_1, B_2, B_3\}$ is a partition of A, is $\{B_1^c, B_2^c, B_3^c\}$ a partition of A? Explain. Consider the possibility that two or more of the elements of $\mathcal{P}$ may be equal.

(c) If $\mathcal{P} = \{B_1, B_2\}$ is a partition of A, $\mathcal{C}_1$ is a partition of B_1, and $\mathcal{C}_2$ is a partition of B_2, and $B_1 \neq B_2$, prove that $\mathcal{C}_1 \cup \mathcal{C}_2$ is a partition of A.

Proofs to Grade

15. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) **Claim.** Let R be an equivalence relation on the set A, and let x, y, and z be elements of A. If $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$.
"Proof." Assume that $x \in y/R$ and $z \in x/R$. Then $y\,R\,x$ and $x\,R\,z$. By transitivity, $y\,R\,z$, so $z \in y/R$. Therefore, if $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$. ■

(b) **Claim.** Let R be an equivalence relation on the set A, and let x, y, and z be elements of A. If $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$.
"Proof." Assume that $x \in y/R$ and assume that $z \in y/R$. Then $y\,R\,x$ and $y\,R\,z$. By symmetry, $x\,R\,y$, and by transitivity, $x\,R\,z$. Therefore, $z \in x/R$. We conclude that if $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$. ■

(c) **Claim.** If $\mathcal{A}$ is a partition of a set A and $\mathcal{B}$ is a partition of a set B, then $\mathcal{A} \cup \mathcal{B}$ is a partition of $A \cup B$.
"Proof."

(i) If $X \in \mathcal{A} \cup \mathcal{B}$, then $X \in \mathcal{A}$, or $X \in \mathcal{B}$. In either case $X \neq \emptyset$.

(ii) If $X \in \mathcal{A} \cup \mathcal{B}$ and $Y \in \mathcal{A} \cup \mathcal{B}$, then $X \in \mathcal{A}$ and $Y \in \mathcal{A}$, or $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, or $X \in \mathcal{B}$ and $Y \in \mathcal{A}$, or $X \in \mathcal{B}$ and $Y \in \mathcal{B}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are partitions, in each case either $X = Y$ or $X \cap Y = \emptyset$.

(iii) Since $\bigcup_{X \in \mathcal{A}} X = A$ and $\bigcup_{x \in \mathcal{B}} X = B$, $\bigcup_{X \in \mathcal{A} \cup \mathcal{B}} X = A \cup B$. ■

★ **(d)** **Claim.** If $\mathcal{B}$ is a partition of A, and if $x\,Q\,y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$, then the relation Q is symmetric.
"Proof." First, $x\,Q\,y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$. Also, $y\,Q\,x$ iff there exists $C \in \mathcal{B}$ such that $y \in C$ and $x \in C$. Therefore, $x\,Q\,y$ iff $y\,Q\,x$. ■

3.4 Ordering Relations

Familiar ordering relations for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ such as "less than," "greater than," and "less than or equal to" are basic to our understanding of number systems but they are not equivalence relations. For instance, $<$ is not reflexive on $\mathbb{R}$ because $3 < 3$ is false, and is not symmetric because $2 < \pi$ is true but $\pi < 2$ is false. The relation $<$ is

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