The terms of a sequence need not be distinct. The first ten terms of the sequence x given by $x_n = ((-1)^n + 1)n^2$ are 0, 8, 0, 32, 0, 72, 0, 128, 0, 200.

If R is an equivalence relation on the set X, then the function from X to X/R that maps each $a \in X$ to the equivalence class of a is called the **canonical map** for the relation R. Recall that in Chapter 3 we used $\bar{a}$ to denote the equivalence class of a. If f is the canonical map for the relation of congruence modulo 5 on $\mathbb{Z}$, the images of -3 and 9 are

$$f(-3) = \overline{-3} = \bar{2} = \{\ldots, -8, -3, 2, 7, 12, \ldots\},$$
$$f(9) = \bar{9} = \bar{4} = \{\ldots, -6, -1, 4, 9, 14, \ldots\}.$$

The equivalence classes under the family name relation L on the set P of all people with family names (see Section 3.1) are sets of people all having the same family name. Under the canonical map f from P to P/L, every person corresponds to his or her equivalence class. Thus f(Charlie Brown) is the set of all people with family name Brown and f(Charlie Brown) = f(Buster Brown). The canonical map is a natural function to consider, and it plays an essential role in the development of many mathematical structures.

Rules of correspondence between equivalence classes have interesting properties. Consider for example the classes $\bar{0}$, $\bar{1}$, $\bar{2}$, and $\bar{3}$ of $\mathbb{Z}_4$ and the rule that $\bar{x}$ in $\mathbb{Z}_4$ corresponds to the equivalence class $[2x]$ in $\mathbb{Z}_{10}$. (Note, for clarity, we use here the bar notation for the equivalence classes of $\mathbb{Z}_4$ and the bracket notation for equivalence classes in $\mathbb{Z}_{10}$.) Under this rule,

$$f(\bar{0}) = [0], \quad f(\bar{1}) = [2], \quad f(\bar{2}) = [4], \quad \text{and} \quad f(\bar{3}) = [6].$$

However, 0 and 4 are in the same class in $\mathbb{Z}_4$, so by the rule $f(\bar{4}) = [8]$. In $\mathbb{Z}_{10}$, however, $[0] \neq [8]$, so the rule assigns two different values to the same element, $\bar{0} = \bar{4}$, of $\mathbb{Z}_4$. Thus, f is not a function. In cases where an object in the domain has more than one representative (for instance, the object $\bar{0}$ can be represented by $\bar{0}$, $\bar{4}$, $\overline{-4}$, $\bar{8}$, $\overline{-8}$, ...) and a supposed function assigns different values that depend on the representative, we say "the function is not well defined," meaning that it is not really a function.

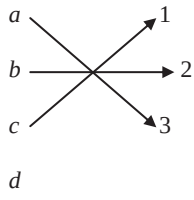
Exercises 4.1

1. Which of the following relations are functions? For those relations that are functions, give the domain and two sets that could be a codomain.

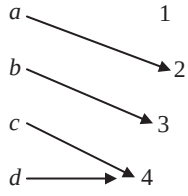
★ **(a)** $\{(0, \triangle), (\triangle, \square), (\square, \cap), (\cap, \cup), (\cup, 0)\}$
(b) $\{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6)\}$
(c) $\{(1, 2), (2, 1)\}$
(d) $\{(x, y) \in \mathbb{R} \times \mathbb{R}: x = \sin y\}$
(e) $\{(x, y) \in \mathbb{N} \times \mathbb{N}: x \leq y\}$
(f) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z}: y^2 = x\}$
(g) $\{(\varnothing, \{\varnothing\}), (\{\varnothing\}, \varnothing), (\varnothing, \varnothing), (\{\varnothing\}, \{\varnothing\})\}$

Copyright 2011 Cengage Learning, Inc. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part.

(h)



(i)



2. Give a relation r from $A = \{5, 6, 7\}$ to $B = \{3, 4, 5\}$ such that
- **(a)** r is not a function.
- **(b)** r is a function, but not a function from A to B.
- **(c)** r is a function from A to B, with $\text{Rng}(r) = B$.
- **(d)** r is a function from A to B, with $\text{Rng}(r) \neq B$.

3. Identify the domain, range, and another possible codomain for each of the following mappings.

★ **(a)** $\left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{1}{x+1}\right\}$

(b) $\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2 + 5\}$

(c) $\{(x, y) \in \mathbb{N} \times \mathbb{N}: y = x + 5\}$

★ **(d)** $\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \tan x\}$

(e) $\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \chi_{\mathbb{N}}(x)\}$

(f) $\left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{e^x + e^{-x}}{2}\right\}$

(g) $\left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{x^2 - 4}{x - 2}\right\}$

(h) $\left\{(x, y) \in \mathbb{Z} \times \mathbb{Z}: y = \dfrac{x^2 - 4}{x - 2}\right\}$

4. Assuming that the domain of each of the following functions is the largest possible subset of $\mathbb{R}$, find the domain and range of

★ **(a)** $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$.

(b) $f(x) = 2x + 5$.

(c) $f(x) = \dfrac{1}{\sqrt{x + \pi}}$.

(d) $f(x) = \sqrt{5 - x}$.

(e) $f(x) = \sqrt{5 - x} + \sqrt{x - 3}$.

(f) $f(x) = \sqrt{x + 2} + \sqrt{-2 - x}$.

Copyright 2011 Cengage Learning, Inc. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part.

★ **5.** **(a)** Let A be the set $\{1, 2, 3, 4\}$ and let R be the relation on A given by $\{(x, y)$: $2x + y$ is prime and not equal to $5\}$. Prove that R is a function with domain A.

(b) Let A be the set $\{1, 2, 3\}$ and let R be the relation on A given by $\{(x, y)$: $3x + y$ is prime$\}$. Prove that R is a function with domain A.

(c) Let R be the relation on $\mathbb{Z}$ given by $\{(x, y)$: $x^2 + y = 2\}$. Prove that R is a function with domain $\mathbb{Z}$.

6. Show that the following relations are not functions on $\mathbb{R}$.

(a) $\{(x, y) \in \mathbb{R} \times \mathbb{R}$: $x^2 = y^2\}$

(b) $\{(x, y) \in \mathbb{R} \times \mathbb{R}$: $x^2 + y^2 = 1\}$

(c) $\{(x, y) \in \mathbb{R} \times \mathbb{R}$: $x = \cos y\}$

(d) $\{(x, y) \in \mathbb{R} \times \mathbb{R}$: $y^2 = \sqrt{x}\}$

7. Let the universe be $\mathbb{R}$ and $A = [1, 3)$. Sketch the graph of

(a) χ_A. **(b)** χ_{A^c}. **(c)** $\chi_{\{1/2\}}$. **(d)** $\chi_{\mathbb{N}}$.

8. Let U be the universe and $A \subseteq U$ with $A \neq \varnothing$, $A \neq U$. Let χ_A be the characteristic function of A.

★ **(a)** What is $\{x \in U$: $\chi_A(x) = 1\}$? **(b)** What is $\{x \in U$: $\chi_A(x) = 0\}$?

(c) What is $\{x \in U$: $\chi_A(x) = 2\}$?

9. Give an example of a sequence x such that

★ **(a)** the range of x is the negative integers.

(b) the terms of x are alternately positive and negative.

☆ **(c)** all terms of x are distinct and between 3 and 4.

(d) the range of x has exactly 3 elements.

10. For the canonical map f: $\mathbb{Z} \to \mathbb{Z}_6$, find

★ **(a)** $f(3)$. **(b)** the image of 6.

(c) a pre-image of $\overline{3}$. **(d)** all pre-images of $\overline{1}$.

11. Which of the following are functions from the indicated domain to the indicated codomain? In each case, we represent an element of the domain as an equivalence class $\bar{x}$, and use the notation $[x]$ for equivalence classes in the codomain. For those relations that are not functions, show that the function is not well defined by naming an equivalence class in the domain that is assigned two different values.

(a) f: $\mathbb{Z}_3 \to \mathbb{Z}_6$ given by $f(\bar{x}) = [x]$

★ **(b)** f: $\mathbb{Z}_6 \to \mathbb{Z}_6$ given by $f(\bar{x}) = [x + 1]$

(c) f: $\mathbb{Z}_3 \to \mathbb{Z}_6$ given by $f(\bar{x}) = [2x]$

(d) f: $\mathbb{Z}_4 \to \mathbb{Z}_6$ given by $f(\bar{x}) = [2x + 1]$

★ **(e)** f: $\mathbb{Z}_3 \to \mathbb{Z}_4$ given by $f(\bar{x}) = [x]$

(f) f: $\mathbb{Z}_4 \to \mathbb{Z}_2$ given by $f(\bar{x}) = [3x]$

12. Explain why the functions $f(x) = \dfrac{9 - x^2}{x + 3}$ and $g(x) = 3 - x$ are not equal.

13. **(a)** Prove that the empty set $\varnothing$ is a function with domain $\varnothing$.

Copyright 2011 Cengage Learning, Inc. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part.

(b) Prove that if $f: A \to B$ and any one of f, A, or $\text{Rng}(f)$ is empty, then all three are empty.

14. Complete the proof of Theorem 4.1.1. That is, prove that if (i) $\text{Dom}(f) = \text{Dom}(g)$ and (ii) for all $x \in \text{Dom}(f)$, $f(x) = g(x)$, then $f = g$.

15. Let S be a relation from A to B. We define two **projection functions** $\pi_1: S \to A$ and $\pi_2: S \to B$ as follows: For all (a, b) in S, $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. In terms of S, find

★ **(a)** $\text{Rng}(\pi_1)$. **(b)** $\text{Rng}(\pi_2)$.

16. A **metric** on a set X is a function $d: X \times X \to \mathbb{R}$ such that for all $a, b, c \in X$,

(i) $d(a, b) \geq 0$.
(ii) $d(a, b) = 0$ iff $a = b$.
(iii) $d(a, b) = d(b, a)$.
(iv) $d(a, b) + d(b, c) \geq d(a, c)$.

Prove that each of the following is a metric for the indicated set.

★ **(a)** $X = \mathbb{N}$, $d(x, y) = |x - y|$

(b) $X = \mathbb{R}$, $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

(c) $X = \mathbb{R} \times \mathbb{R}$, $d((x, y), (z, w)) = \sqrt{(x - z)^2 + (y - w)^2}$

(d) $X = \mathbb{R} \times \mathbb{R}$, $d((x, y), (z, w)) = |x - z| + |y - w|$

17. Suppose that set A has m elements and set B has n elements. We have seen that $A \times B$ has mn elements and that there are 2^{mn} relations from A to B. Find the number of relations from A to B that are

(a) functions from A to B.
(b) functions with one element in the domain.
★ **(c)** functions with two elements in the domain.
(d) functions whose domain is a subset of A.

18. **(a)** Let f be a function from A to B. Define the relation T on A by $x \; T \; y$ iff $f(x) = f(y)$. Prove that T is an equivalence relation on A.

(b) In the case when $f: \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$, describe the equivalence class of 0; of 2; of 4.

(c) In the case when $f: \mathbb{R} \to \mathbb{R}$ is the cosine function, describe the equivalence class of 0; of $\pi/2$; of $\pi/4$.

Proofs to Grade

19. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) **Claim.** The functions f and g are equal, where f and g are given by

$$f(x) = \frac{x}{|x|} \text{ and } g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

"Proof." Let x be a real number. If x is positive, then $\frac{x}{|x|} = \frac{x}{x} = 1$, so $f(x) = g(x)$. If x is negative, then $\frac{x}{|x|} = \frac{x}{-x} = -1$, so $f(x) = g(x)$. In every case, $f(x) = g(x)$, so $f = g$. ■

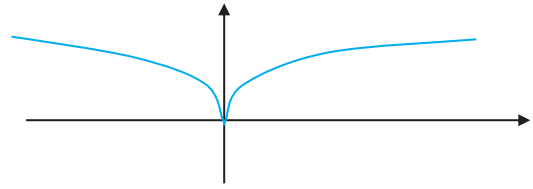
Copyright 2011 Cengage Learning, Inc. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part.

(b) **Claim.** The functions $f(x) = 1 + \frac{1}{x}$ and $g(x) = \frac{x+1}{x}$ are equal.
"Proof." The domain of each function is assumed to be the largest possible subset of $\mathbb{R}$. Thus $\text{Dom}(f) = \text{Dom}(g) = \mathbb{R} - \{0\}$. For every $x \in \mathbb{R} - \{0\}$ we have

$$f(x) = 1 + \frac{1}{x} = \frac{x}{x} + \frac{1}{x} = \frac{x+1}{x} = g(x).$$

Therefore, by Theorem 4.1.1, $f = g$. ■

(c) **Claim.** The relation $x^2 = y^3$ defines a function from $\mathbb{R}$ to $\mathbb{R}$.
"Proof." The graph of the relation $x^2 = y^3$ is given here:



Since no vertical line crosses the graph more than once, $x^2 = y^3$ defines a function. ■

(d) **Claim.** If $h: A \to B$ and $g: C \to D$, then $h \cup g: A \cup C \to B \cup D$.
"Proof." Suppose $(x, y) \in h \cup g$ and $(x, z) \in h \cup g$. Then $(x, y) \in h$ or $(x, y) \in g$, and $(x, z) \in h$ or $(x, z) \in g$. If $(x, y) \in h$ and $(x, z) \in h$, then $y = z$. Otherwise, $(x, y) \in g$ and $(x, z) \in g$; so again $y = z$. Therefore, $h \cup g$ is a function. Thus, we have that $\text{Dom}(h \cup g) = \text{Dom}(h) \cup \text{Dom}(g) = A \cup C$, so $h \cup g: A \cup C \to B \cup D$. ■

(e) **Claim.** The rule that assigns to each equivalence class $\bar{x}$ in $\mathbb{Z}_4$ the class $[x + 1]$ in $\mathbb{Z}_2$ is a function.
"Proof." Suppose $(\bar{x}, [y])$ and $(\bar{x}, [z])$ are two ordered pairs in the relation determined by the rule. We must show that $[y] = [z]$. According to the rule, $[y] = [x_1 + 1]$ and $[z] = [x_2 + 1]$ for some x_1, x_2 in the class $\bar{x}$. Since x_1 and x_2 are in the same equivalence class (mod 4), $x_1 - x_2 = 4k$ for some integer k. Then $(x_1 + 1) - (x_2 + 1) = 4k = 2(2k)$, so $x_1 + 1$ and $x_2 + 1$ are in the same equivalence class (mod 2). Therefore $[y] = [z]$. ■

4.2 Constructions of Functions

This section discusses several methods for constructing new functions from given ones. You have already seen the operations of composition and inversion of relations in Chapter 3. Since every function is a relation, these operations are performed on functions in the same way.

Copyright 2011 Cengage Learning, Inc. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part.