The graph of the function f in Figure 4.2.5 is increasing on the interval $[0, 2]$ and decreasing on the intervals $(-\infty, 0]$ and $[2, \infty)$. This seems clear from the graph, but remember that looking at its graph does not constitute a proof because we cannot see the entire graph or all the details of its behavior. We give two examples of proofs of these properties.

Example. Prove that the function f of Figure 4.2.5 is increasing on the interval $[0, 2]$.

Proof. Let $x, y \in [0, 2]$ and suppose $x < y$. Then $g(x) = x^2$ and $g(y) = y^2$. Since $x < y$ and $x \geq 0$, $x^2 \leq xy$. Since $x < y$ and $y > 0$, $xy < y^2$. Therefore, $x^2 < y^2$. Thus $g(x) < g(y)$, so g is increasing on $[0, 2]$. ■

Example. Let $f(x) = 2 + \dfrac{1}{x}$. Prove that f is decreasing for $x > 0$.

Proof. Suppose $0 < x < y$. Then $f(x) = 2 + \dfrac{1}{x}$ and $f(y) = 2 + \dfrac{1}{y}$. Since x and y are positive and $x < y$ we have $\dfrac{1}{y} < \dfrac{1}{x}$. Thus $2 + \dfrac{1}{y} < 2 + \dfrac{1}{x}$; that is, $f(x) > f(y)$. Therefore f is decreasing on $(0, \infty)$. ■

Exercises 4.2

1. Find $f \circ g$ and $g \circ f$ for each pair of functions f and g. Use the understood domains for f and g.

★ **(a)** $f(x) = 2x + 5, \quad g(x) = 6 - 7x$

(b) $f(x) = x^2 + 2x, \quad g(x) = 2x + 1$

★ **(c)** $f(x) = \sin x, \quad g(x) = 2x^2 + 1$

(d) $f(x) = \tan x, \quad g(x) = \sin x$

★ **(e)** $f(x) = \{(t, r), (s, r), (k, l)\}, \quad g(x) = \{(k, s), (t, s), (s, k)\}$

(f) $f(x) = \{(1, 3), (2, 6), (3, 5), (4, 2), (5, 2)\}$,
$g(x) = \{(1, 5), (2, 3), (3, 7), (4, 3), (5, 4)\}$

(g) $f(x) = \dfrac{x + 1}{x + 2}, \quad g(x) = x^2 + 1$

(h) $f(x) = 3x + 2, \quad g(x) = |x|$

★ **(i)** $f(x) = \begin{cases} x + 1 & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases}, \quad g(x) = \begin{cases} 2x & \text{if } x \leq -1 \\ -x & \text{if } x > -1 \end{cases}$

(j) $f(x) = \begin{cases} 2x + 3 & \text{if } x < 3 \\ x^2 & \text{if } x \geq 3 \end{cases}, \quad g(x) = \begin{cases} 7 - 2x & \text{if } x \leq 2 \\ x + 1 & \text{if } x > 2 \end{cases}$

☆ **2.** Find the domain and range of each composite in Exercise 1.

3. Give two different examples of

★ **(a)** a pair of functions f and g such that $(f \circ g)(x) = (3x + 7)^2$.

(b) a pair of functions f and g such that $(f \circ g)(x) = \sqrt{2x^2 - 5}$.

(c) a pair of functions f and g such that $(f \circ g)(x) = \sin|2x + 4|$.

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4. Let $\mathbb{Z}_8 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7},\}$ and $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$. Define $f: \mathbb{Z}_8 \to \mathbb{Z}_4$, $g: \mathbb{Z}_4 \to \mathbb{Z}_8$, $h: \mathbb{Z}_8 \to \mathbb{Z}_8$, and $k: \mathbb{Z}_4 \to \mathbb{Z}_4$, as follows: $f(\overline{x}) = [x + 2]$, $g([x]) = \overline{2x}$, $h(\overline{x}) = \overline{2x + 4}$, and $k([x]) = [2x + 2]$). By comparing images, verify the following equalities.

(a) $(k \circ k)(x) = [2]$ for all x in $\mathbb{Z}_4$.
(b) $(g \circ f)(x) = h(x)$ for all x in $\mathbb{Z}_8$.
(c) $(f \circ g)(x) = k(x)$ for all x in $\mathbb{Z}_4$.
(d) $(h \circ (h \circ h))(x) = \overline{4}$ for all x in $\mathbb{Z}_8$.

5. For which of the following functions f is the relation f^{-1} a function? When f^{-1} is a function, write an explicit expression for $f^{-1}(x)$. Use the understood domain for each function.

★ **(a)** $f(x) = 5x + 2$ **(b)** $f(x) = 2x^2 + 1$ ★ **(c)** $f(x) = \dfrac{x+1}{x+2}$

(d) $f(x) = \sin x$ ★ **(e)** $f(x) = e^{x+3}$ **(f)** $f(x) = \dfrac{1-x}{-x}$

(g) $f(x) = \dfrac{1}{1-x}$ **(h)** $f(x) = -x + 3$ **(i)** $f(x) = \dfrac{-x}{3x-4}$

6. Prove the remaining part of Theorem 4.2.3: if $f: A \to B$, then $I_B \circ f = f$.

7. Prove the remaining part of Theorem 4.2.4: if $f: A \to B$ with $\text{Rng}(f) = C$, and if f^{-1} is a function, then $f \circ f^{-1} = I_C$.

8. Let $f(x) = 4 - 3x$ with domain $\mathbb{R}$ and $A = \{1, 2, 3, 4\}$. Sketch the graphs of the functions $f|_A$, $f|_{[-1,\,3]}$, $f|_{(2,\,4]}$, and $f|_{\{6\}}$. What is the range of $f|_{\mathbb{N}}$?

9. Describe two extensions of f with domain $\mathbb{R}$ for the function

★ **(a)** $f = \{(x, y) \in \mathbb{N} \times \mathbb{N}: y = x^2\}$.
(b) $f = \{(x, y) \in \mathbb{N} \times \mathbb{N}: y = 3\}$.
(c) $f = \{(x, y) \in [-1, 1] \times [-1, 1]: y = -x\}$.

10. Prove that, if f and g are functions, then $f \cap g$ is a function by showing that $f \cap g = g|_A$ where $A = \{x: g(x) = f(x)\}$.

11. Prove Theorem 4.2.5.

12. Let f be a function with domain D, and let g be an extension of f with domain A. Then by definition, $f = g|D$ and $D \subseteq A$. Let i be the inclusion mapping from D to A given by $i(x) = x$ for all $x \in D$. Prove that $f = g \circ i$.

☆ **13.** Let $h: A \to B$, $g: C \to D$ and suppose $E = A \cap C$. Prove $h \cup g$ is a function from $A \cup C$ to $B \cup D$ if and only if $h|_E = g|_E$.

14. For each pair of functions h and g, determine whether $h \cup g$ is a function. In each case sketch a graph of $h \cup g$.

★ **(a)** $h: (-\infty, 0] \to \mathbb{R}$, $h(x) = 3x + 4$
$g: (0, \infty) \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$

(b) $h: [-1, \infty) \to \mathbb{R}$, $h(x) = x^2 + 1$
$g: (-\infty, -1] \to \mathbb{R}$, $g(x) = x + 3$

(c) $h: (-\infty, 1] \to \mathbb{R}$, $h(x) = |x|$
$g: [0, \infty) \to \mathbb{R}$, $g(x) = 3 - |x - 3|$

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(d) $h\colon (-\infty, 2] \to \mathbb{R}, \quad h(x) = \cos x$
$g\colon [2, \infty) \to \mathbb{R}, \quad g(x) = x^2$

(e) $h\colon (-\infty, 3) \to \mathbb{R}, \quad h(x) = 3 - x$
$g\colon (0, \infty) \to \mathbb{R}, \quad g(x) = x + 1$

15. Let $f\colon A \to B$ and $g\colon C \to D$. Define $f \times g = \{((a, c), (b, d))\colon (a, b) \in f$ and $(c, d) \in g\}$.

(a) Prove that $f \times g$ is a function from $A \times C$ to $B \times D$.

(b) For $(a, c) \in A \times C$, write $(f \times g)(a, c)$ in terms of $f(a)$ and $g(c)$.

16. Prove each of these statements.

★ **(a)** f is increasing on $\mathbb{R}$, where $f(x) = 3x - 7$.

(b) g is decreasing on $\mathbb{R}$, where $g(x) = 2 - 5x$.

(c) h is increasing on $[0, \infty)$, where $h(x) = x^2$.

★ **(d)** f is increasing on $(-3, \infty)$, where $f(x) = \dfrac{x - 1}{x + 3}$.

17. Prove or give a counterexample:

(a) If f is a linear function with positive slope, f is increasing on $\mathbb{R}$.

(b) If f and g are decreasing functions on an interval I and $f \circ g$ is defined on I, then $f \circ g$ is decreasing on I.

(c) If f and g are decreasing functions on I and $f \circ g$ is defined on I, then $f \circ g$ is increasing on I.

★ **(d)** If $\mathrm{Dom}(f) = \mathbb{R}$ and f is increasing on the intervals $[-2, -1]$ and $[1, 2]$, then f is increasing on $[-2, 2]$.

(e) If f is decreasing on $(-\infty, 0)$ and decreasing on $[0, \infty)$, then f is decreasing on $\mathbb{R}$.

18. Let $f_1\colon \mathbb{R} \to \mathbb{R}$ and $f_2\colon \mathbb{R} \to \mathbb{R}$. Define the **pointwise sum** $f_1 + f_2$ and **pointwise product** $f_1 \cdot f_2$ as follows:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) \text{ for all } x \in \mathbb{R} \text{ and}$$
$$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) \text{ for all } x \in \mathbb{R}.$$

☆ **(a)** Prove that $f_1 + f_2$ and $f_1 \cdot f_2$ are functions with domain $\mathbb{R}$.

(b) Let $f(x) = 2x + 5$, $g(x) = 6 - 7x$, and $h(x) = 3x^2 - 7x + 2$. Compute $(f + g)(x)$, $(f \cdot g)(x)$, $(f + h)(x)$, and $(g \cdot h)(x)$.

(c) Prove or disprove: If f and g are increasing on $\mathbb{R}$, and h is decreasing on $\mathbb{R}$, then $(f + g) + h$ is increasing on $\mathbb{R}$.

19. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Define the **scalar product** cf:

$$cf(x) = c \cdot f(x) \text{ for all } x \in \mathbb{R}.$$

Prove that cf is a function with domain $\mathbb{R}$.

Proofs to Grade

20. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** Let $f\colon A \to B$. If f^{-1} is a function, then $f^{-1} \circ f = I_A$.
"Proof." Suppose $(x, y) \in f^{-1} \circ f$. Then there is z such that $(x, z) \in f$ and $(z, y) \in f^{-1}$. But this means that $(z, x) \in f^{-1}$ and $(z, y) \in f^{-1}$. Since f^{-1} is a function, $x = y$. Hence $(x, y) \in f^{-1} \circ f$

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implies $(x, y) \in I_A$; that is, $f^{-1} \circ f \subseteq I_A$. Now suppose $(x, y) \in I_A$. Since $A = \text{Dom}(f)$, there is $w \in B$ such that $(x, w) \in f$. Hence $(w, x) \in f^{-1}$. But $(x, y) \in I_A$ implies $x = y$ and so $(w, y) \in f^{-1}$. But from $(x, w) \in f$ and $(w, y) \in f^{-1}$, we have $(x, y) \in f^{-1} \circ f$. This shows $I_A \subseteq f^{-1} \circ f$. Therefore, $I_A = f^{-1} \circ f$. ■

(b) **Claim.** If f and f^{-1} are functions on A, and $f \circ f = f$, then $f = I_A$.
"Proof." Suppose $f: A \to A$ and $f^{-1}: A \to A$. Since $f = f \circ f$, $f^{-1} \circ f = f^{-1} \circ (f \circ f)$. By associativity, we have $f^{-1} \circ f = (f^{-1} \circ f) \circ f$. This gives $I_A = I_A \circ f$. Since $I_A \circ f = f$, by cancellation we have $I_A = f$. ■

(c) **Claim.** If f, g, and f^{-1} are functions on A, then $g = f^{-1} \circ (g \circ f)$.
"Proof." Using associativity and Theorems 4.2.3 and 4.2.4,

$$f^{-1} \circ (g \circ f) = f^{-1} \circ (f \circ g) = (f^{-1} \circ f) \circ g = I_A \circ g = g.$$ ■

(d) **Claim.** If $f'(x) > 0$ on an open interval (a, b), then f is increasing on (a, b).
"Proof." Assume that $f'(x) > 0$ on the interval (a, b). Suppose x_1 and x_2 are in (a, b) and $x_1 < x_2$. We must show that $f(x_1) < f(x_2)$. We know from calculus that since f is differentiable on (a, b), it is continuous on $[x_1, x_2] \subseteq (a, b)$, and differentiable on (x_1, x_2). By the Mean Value Theorem, there exists c in (x_1, x_2) such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

Therefore, $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. By hypothesis $f'(c) > 0$, and $x_2 - x_1 > 0$ since $x_1 < x_2$. Therefore, $f(x_2) - f(x_1) > 0$. We conclude that $f(x_1) < f(x_2)$. ■

4.3 Functions That Are Onto; One-to-One Functions

The definition of a function $f: A \to B$ is stated in terms of conditions on the first coordinates of f. Two important properties of functions are defined by requiring additional conditions on the second coordinates of f.

DEFINITION A function $f: A \to B$ is **onto B** (or is a **surjection**) iff $\text{Rng}(f) = B$. When f is a surjection, we write $f: A \xrightarrow{\text{onto}} B$

For $f: A \to B$, it is always the case that $\text{Rng}(f) \subseteq B$, so a function always maps to its codomain. We say f maps onto its codomain when the codomain is $\text{Rng}(f)$.

From our discussion in Section 4.1, we know that the functions

$$f: \mathbb{N} \to \mathbb{R}, \text{ where } f(n) = 2n$$
$$\text{and } g: \mathbb{N} \to E^+, \text{ where } g(n) = 2n$$

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