Proof. Parts (a) and (b) are left as Exercise 7.

(c) Suppose $h\colon A \xrightarrow{1-1} C$, $g\colon B \xrightarrow{1-1} D$, $A \cap B = \varnothing$, and $C \cap D = \varnothing$. Then by Theorem 4.2.5, $h \cup g$ is a function with domain $A \cup B$.

Suppose $x, y \in A \cup B$. Assume $(h \cup g)(x) = (h \cup g)(y)$.

(i) If $x, y \in A$, then $h(x) = (h \cup g)(x) = (h \cup g)(y) = h(y)$. Since h is one-to-one, $x = y$.

(ii) If $x, y \in B$, then by a similar argument, $g(x) = g(y)$, and g is one-to-one; so $x = y$.

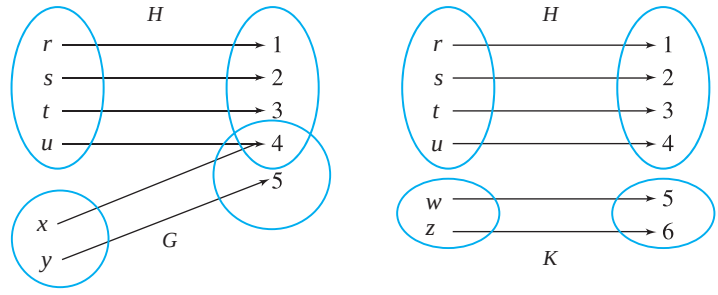
(iii) Suppose $x \in A$ and $y \in B$. Then $h(x) = g(y)$ and $h(x) \in C$ and $g(y) \in D$. But $C \cap D = \varnothing$. This case is impossible.

(iv) Similarly, $x \in B$ and $y \in A$ is impossible.

In every possible case, $x = y$. Therefore, $h \cup g$ is one-to-one. ■

Examples. Let H be the function on $\{r, s, t, u\}$ that sends r to 1, s to 2, t to 3, and u to 4. Then H is one-to-one and onto $\{1, 2, 3, 4\}$. The function $G = \{(x, 4), (y, 5)\}$ maps $\{x, y\}$ one-to-one and onto $\{4, 5\}$. Since the domains are disjoint, $H \cup G$ is a function. By Theorem 4.3.5 (b), $H \cup G$ is onto $\{1, 2, 3, 4\} \cup \{4, 5\} = \{1, 2, 3, 4, 5\}$. Notice that we cannot apply part (c) of Theorem 4.3.5, because the ranges of H and G are not disjoint. See Figure 4.3.3(a).

If we let $K = \{(w, 5), (z, 6)\}$, then the domains of H and K are disjoint and the ranges of H and K are disjoint. In this case, $H \cup K$ is a function that maps $\{r, s, t, u, w, z\}$ one-to-one and onto $\{1, 2, 3, 4, 5, 6\}$. See Figure 4.3.3(b).



(a) $H \cup G$ (b) $H \cup K$

Figure 4.3.3

Exercises 4.3

1. Which of the following functions map onto their indicated codomains? Prove each of your answers.

★ **(a)** $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}x + 6$

(b) $f\colon \mathbb{Z} \to \mathbb{Z}$, $f(x) = -x + 1{,}000$

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★ **(c)** $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N},\ f(x) = (x, x)$
(d) $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
(e) $f: \mathbb{R} \to \mathbb{R},\ f(x) = \sqrt{x^2 + 5}$
(f) $f: \mathbb{R} \to \mathbb{R},\ f(x) = 2^x$
(g) $f: \mathbb{R} \to \mathbb{R},\ f(x) = \sin x$
(h) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ f(x, y) = x - y$
(i) $f: \mathbb{R} \to [-1, 1],\ f(x) = \cos x$
(j) $f: \mathbb{R} \to [1, \infty),\ f(x) = x^2 + 1$
★ **(k)** $f: [2, 3) \to [0, \infty),\ f(x) = \dfrac{x - 2}{3 - x}$
(l) $f: (1, \infty) \to (1, \infty),\ f(x) = \dfrac{x}{x - 1}$

☆ **2.** Which of the functions in Exercise 1 are one-to-one? Prove each of your answers.

3. For each function, determine whether the function maps onto the given codomain and whether it is one-to-one. Prove your answers.
(a) The identity function I_A from A to A.
(b) The canonical map from $\mathbb{Z}$ to $\mathbb{Z}_5$.
(c) The greatest integer function int: $\mathbb{R} \to \mathbb{Z}$.
(d) The sequence a: $\mathbb{N} \to \mathbb{N}$ whose nth term is $a_n = 2^n$.

4. Let $A = \{1, 2, 3, 4\}$. Describe a codomain B and a function $f: A \to B$ such that f is
★ **(a)** onto B but not one-to-one.
(b) one-to-one but not onto B.
(c) both one-to-one and onto B.
(d) neither one-to-one nor onto B.

5. Prove that if $f: A \xrightarrow{\text{onto}} B$, and $g: B \xrightarrow{\text{onto}} C$, then $g \circ f: A \xrightarrow{\text{onto}} C$ (Theorem 4.3.1).

6. Prove that if $f: A \to B$, $g: B \to C$, and $g \circ f: A \xrightarrow{1-1} C$, then $f: A \xrightarrow{1-1} B$ (Theorem 4.3.4).

7. Prove parts (a) and (b) of Theorem 4.3.5.

8. Find sets A, B, C, and functions $f: A \to B$ and $g: B \to C$ such that
★ **(a)** f is onto B, but $g \circ f$ is not onto C.
(b) g is onto C, but $g \circ f$ is not onto C.
(c) $g \circ f$ is onto C, but f is not onto B.
(d) f is one-to-one, but $g \circ f$ is not one-to-one.
★ **(e)** g is one-to-one, but $g \circ f$ is not one-to-one.
(f) $g \circ f$ is one-to-one, but g is not one-to-one.

9. Prove that
(a) $f(x) = \begin{cases} 2 - x & \text{if } x \le 1 \\ \dfrac{1}{x} & \text{if } x > 1 \end{cases}$ is one-to-one but not onto $\mathbb{R}$.

(b) $f(x) = \begin{cases} x + 4 & \text{if } x \le -2 \\ -x & \text{if } -2 < x < 2 \\ x - 4 & \text{if } x \ge 2 \end{cases}$ is onto $\mathbb{R}$ but not one-to-one.

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★ **(c)** $f(x) = \begin{cases} \dfrac{x-2}{x+4} & \text{if } x \neq -4 \\ 1 & \text{if } x = -4 \end{cases}$ is one-to-one and onto $\mathbb{R}$.

(d) $f(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$ is neither one-to-one nor onto $\mathbb{R}$.

10. Prove that if the real-valued function f is increasing (or decreasing), then f is one-to-one.

11. ★**(a)** Let $f: \mathbb{Z}_4 \to \mathbb{Z}_8$ be given by $f(\bar{x}) = [2x]$, for each $\bar{x} \in \mathbb{Z}_4$. Prove that f is one-to-one but not onto $\mathbb{Z}_8$.

(b) Let $f: \mathbb{Z}_4 \to \mathbb{Z}_2$ be given by $f(\bar{x}) = [3x]$, for each $x \in \mathbb{Z}_4$. Prove that f is onto $\mathbb{Z}_2$, but not one-to-one.

(c) Let $f: \mathbb{Z}_6 \to \mathbb{Z}_6$ be given by $f(\bar{x}) = \overline{x+1}$, for each $\bar{x} \in \mathbb{Z}_6$. Prove that f is a bijection.

(d) Let $f: \mathbb{Z}_4 \to \mathbb{Z}_4$ be given by $f(\bar{x}) = \overline{2x}$. Prove that f is not one-to-one and not onto $\mathbb{Z}_4$.

12. Give two examples of a sequence x of natural numbers (i.e., a function with domain $\mathbb{N}$ and range that is a subset of $\mathbb{N}$) such that

(a) x is neither one-to-one nor onto $\mathbb{N}$.

★ **(b)** x is one-to-one and onto $\mathbb{N}$.

(c) x is one-to-one and not onto $\mathbb{N}$.

(d) x is onto $\mathbb{N}$ and not one-to-one.

13. Suppose the set A has m elements and the set B has n elements. By Exercise 17 in Section 4.1, there are 2^{mn} relations from A to B and n^m functions from A to B.

(a) If $m < n$, find the number of one-to-one functions from A to B.

(b) If $m = n$, find the number of one-to-one functions from A to B.

☆ **(c)** If $m > n$, find the number of one-to-one functions from A to B.

(d) If $m < n$, find the number of functions from A onto B.

(e) If $m = n$, find the number of functions from A onto B.

★ **(f)** If $m = n + 1$, find the number of functions from A onto B.

(g) If $m = n$, find the number of one-to-one correspondences from A onto B.

Proofs to Grade

14. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ **(a)** **Claim.** The function $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x, y) = 2x - 3y$ is a surjection.

"Proof." Suppose $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then $x \in \mathbb{R}$, so $2x \in \mathbb{R}$. Also, $y \in \mathbb{R}$, so $3y \in \mathbb{R}$. Therefore $2x - 3y \in \mathbb{R}$. Thus $f(x, y) \in \mathbb{R}$, so f is a surjection. ■

☆ **(b)** **Claim.** The function $f: [1, \infty) \to (0, \infty)$ defined by $f(x) = \frac{1}{x}$ maps onto $(0, \infty)$.

"Proof." Suppose $w \in (0, \infty)$. Choose $x = \frac{1}{w}$. Then $f(x) = \dfrac{1}{\frac{1}{w}} = w$.

Therefore the function f is onto $(0, \infty)$. ■

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(c) **Claim.** If $f: A \xrightarrow{\text{onto}} B$ and $g: B \xrightarrow{\text{onto}} C$, then $g \circ f: A \to C$ maps onto C. (Theorem 4.3.1)
"Proof." Suppose $a \in A$. Then $f(a) \in B$. Since $f(a) \in B$, $g(f(a)) \in C$. Therefore, $(g \circ f)(a) = g(f(a)) \in C$, so $g \circ f$ is onto C. ■

★ **(d)** **Claim.** The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 7$ is onto $\mathbb{R}$.
"Proof." Suppose f is not onto $\mathbb{R}$. Then there exists $b \in \mathbb{R}$ with $b \notin \text{Rng}(f)$. Thus for all real numbers x, $b \neq 2x + 7$. But $a = \frac{1}{2}(b - 7)$ is a real number and $f(a) = b$. This is a contradiction. Thus f is onto $\mathbb{R}$. ■

(e) **Claim.** Let I be the interval $(0, 1)$. The function $f: I \times I \to I$ given by $f(x, y) = x^y$ is a surjection.
"Proof." Let $t \in I$. Then $0 < t < 1$, so $0 < t^2 < t < 1$, so $t^2 \in I$. Choose $x = t^2$ and $y = \frac{1}{2} \in I$. Then $f(x, y) = x^y = (t^2)^{1/2} = t$. Therefore, $I \subseteq \text{Rng}(f)$, so the function f is onto I. ■

(f) **Claim.** If $f: A \xrightarrow{1-1} B$ and $g: B \xrightarrow{1-1} C$, then $g \circ f: A \xrightarrow{1-1} C$ (Theorem 4.3.3).
"Proof." We must show that if (x, y) and (z, y) are elements of $g \circ f$, then $x = z$. If $(x, y) \in g \circ f$, then there is $u \in B$ such that $(x, u) \in f$ and $(u, y) \in g$. If $(z, y) \in g \circ f$, then there is $v \in B$ such that $(z, v) \in f$ and $(v, y) \in g$. However, $(u, y) \in g$ and $(v, y) \in g$ imply $u = v$ since g is one-to-one. Then $(x, u) \in f$ and $(z, v) \in f$ and $u = v$; therefore, $x = z$, since f is one-to-one. Hence (x, y) and (z, y) in $g \circ f$ imply $x = z$. Therefore, $g \circ f$ is one-to-one. ■

(g) **Claim.** The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x + 7$ is one-to-one.
"Proof." Suppose x_1 and x_2 are real numbers with $f(x_1) \neq f(x_2)$. Then $2x_1 + 7 \neq 2x_2 + 7$ and thus $2x_1 \neq 2x_2$. Hence $x_1 \neq x_2$, which shows that f is one-to-one. ■

(h) **Claim.** The function f in part (e) is an injection.
"Proof." Suppose (x, y) and (x, z) are in $I \times I$ and $f(x, y) = f(x, z)$. Then $x^y = x^z$. Dividing by x^z, we have $x^{y-z} = x^0 = 1$. Since $x \neq 1$ and $x^{y-z} = 1$, $y - z$ must be 0. Therefore, $y = z$. This shows that $(x, y) = (x, z)$, so f is an injection. ■

(i) **Claim.** The function f in part (e) is not an injection.
"Proof." Both $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{4}, \frac{1}{4}\right)$ are in $I \times I$. But

$$f\left(\frac{1}{4}, \frac{1}{4}\right) = \left(\frac{1}{4}\right)^{\frac{1}{4}} = \left(\left(\frac{1}{2}\right)^2\right)^{\frac{1}{4}} = \left(\frac{1}{2}\right)^{\frac{1}{2}} = f\left(\frac{1}{2}, \frac{1}{2}\right).$$ ■

4.4 One-to-One Correspondences and Inverse Functions

In this section we consider functions that have both the desirable properties of being one-to-one and mapping onto their codomains. The key role played by these functions in succeeding chapters suggests their importance in advanced mathematics.

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