Math 108, Fall 2026.
Oct. 23, 2026.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): _____

ID#: _____

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work to receive full credit.* Clarity of your solutions may be a factor when determining credit. Electronic devices, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions: proper interpretation of exam questions is a part of the exam.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. Assume that propositions P and Q are true, while R and S are false. For each of the following logical expressions, determine whether it is true or false.

(a) $P \wedge Q \wedge (\sim R)$

True, as all $P, Q, \sim R$ are true.

(b) $\sim (Q \wedge R \wedge S)$

True, as $Q \wedge R \wedge S$ is false.

(c) $(P \Longrightarrow Q) \wedge (R \Longrightarrow S)$

True $P \Rightarrow Q$ is true as they are both true and $R \Rightarrow S$ is, further as R is false.

(d) $(P \vee R) \Longleftrightarrow (Q \wedge S)$

False $P \vee R$ is true but $Q \wedge S$ is false.

2. For each statement below, determine whether it is true or false for the universe $\mathbb{N} = \{1, 2, 3, \ldots\}$. Justify your answers.

(a) $(\exists x)(\exists y)(x + 2y = 11)$

True. $x = 1$, $y = 5$, for example.

(b) $(\forall y)(\exists x)(x + 2y = 11)$

False. For $y = 6$, there is no x such that $x + 2y = 11$, as $x + 2y = x + 12 > 11$.

(c) $(\exists x)(\forall y)(xy \leq x^2 + 1)$

False. Negation

$(\forall x)(\exists y)(xy > x^2 + 1)$

is true: pick an x, choose $y = x + 2$; then

$$xy = x(x+2) = x^2 + 2x > x^2 + 1.$$

3. Suppose x is a real number. Prove that $x^2 - x - 6 < 0$ implies $|x| < 3$. (*Hint.* You may use contrapositive, and then handle the absolute value by two cases.)

Contrapositive:

$|x| \geq 3 \implies x^2 - x - 6 \geq 0$

Case 1. $x \geq 3$

Then $x^2 - x - 6 = (x-3)(x+2) \geq 0$

as both factors are nonnegative.

Case 2. $x \leq -3$

Then $x^2 - x - 6 = (x-3)(x+2) \geq 0$

as both factors are negative.

4. Assume that A, B, C are arbitrary subsets of $\mathbb{N} = \{1, 2, 3, \ldots\}$.

(a) Prove: If $A \subseteq B$ and $B \cap C = \emptyset$, then $A \cap C = \emptyset$. (You may use proof by contradiction.)

Assume $A \cap C \neq \emptyset$, Then there exists an $x \in A \cap C$. Then $x \in A$ and $x \in C$, and then, as $A \subseteq B$, $x \in B$ and $x \in C$. But then $B \cap C \neq \emptyset$ in contradiction with the assumption.

(b) Give a counterexample: If $C \subseteq A \cup B$, then either $C \subseteq A$ or $C \subseteq B$.

$A = \{1\}$, $B = \{2\}$, $C = A \cup B = \{1, 2\}$

Clearly $C \subseteq A \cup B$ and $C \not\subseteq A$ and $C \not\subseteq B$.

(c) Prove: $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

Assume $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$, Then either $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$, that is either $X \subseteq A$ or $X \subseteq B$. In either case, $X \subseteq A \cup B$, so that $X \in \mathcal{P}(A \cup B)$.

5. Prove by mathematical induction that, for every natural number $n \geq 2$,

$$3^n > n2^n .$$

$(n = 2)$ $3^2 > 2 \cdot 2^2$ ✓

$(n \to n+1)$ Assume $3^n > n2^n$.

We need: $3^{n+1} > (n+1)2^{n+1}$ (*).

By I.H., we have $3^{n+1} = 3 \cdot 3^n > 3n2^n$,

So the desired inequality holds if

$$3n2^n \geq (n+1)2^{n+1}$$

$\Updownarrow$ (divide by 2^n)

$$3n \geq 2(n+1)$$

$\Updownarrow$

$$3n \geq 2n + 2$$

$\Updownarrow$

$$n \geq 2,$$

So, we have shown that

$$3^{n+1} > 3n2^n \geq (n+1)2^{n+1},$$

which establishes (*).