Math 135A, Fall 2017.
Dec. 1, 2017.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ______________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. Unless directed to do so, do *not* evaluate complicated expressions to give the result as a fraction or a decimal number. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. A random variable X has density function

$$f(x) = \begin{cases} c(1+3x^2) & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

A random variable Y is independent of X and uniform on $[0,1]$.
(a) Determine the constant c.

$$c \int_0^1 (1+3x^2) = 1$$

$$c(1+1) = 1 \quad , \quad \underline{\underline{c = 1/2}}$$

(b) Compute $E(X)$.

$$EX = \frac{1}{2}\int_0^1 x(1+3x^2)\,dx = \frac{1}{2}\left(\frac{1}{2}+\frac{3}{4}\right) = \underline{\underline{\frac{5}{8}}}$$

(c) Determine the probability $P(Y \leq X)$.

$$P(Y \leq X) = \int_0^1 dx \int_0^x \frac{1}{2}(1+3x^2)\,dy$$

By independence

$$f(x,y) = f_X(x)\, f_Y(y) = \begin{cases} \frac{1}{2}(1+3x^2) & x, y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_0^1 \frac{1}{2} x(1+3x^2)\,dx = \underline{\underline{\frac{5}{8}}} \quad (\text{see (b)})$$

2. A deck of 11 cards contains 4 red and 7 green cards. Shuffle this deck, then deal 4 cards to Alice and 4 cards to Bob (without any replacement, of course). Let X be the number of red cards that Alice gets, and Y the number of red cards that Bob gets.
(a) Write down the joint probability mass function of X and Y. (Write a formula rather than a table.)

$$P(X=i, Y=j) = \frac{\binom{4}{i}\binom{4}{j}\binom{3}{4-i-j}}{\binom{11}{4}}$$

$0 \le i \le 4$, $0 \le j \le 4$, $1 \le i+j \le 4$

positions of red cards in shuffled deck

(b) Are X and Y independent?

No. $P(X=4) \neq 0$, $P(Y=4) \neq 0$, but

$P(X=4, Y=4) = 0 \neq P(X=4)\, P(Y=4)$

(c) Compute the conditional probability $P(X = 1|Y = 1)$. Give the result as a simple fraction.

$$P(X=1, Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)}$$

$$= \frac{\cancel{\binom{4}{1}}\binom{4}{1}\binom{3}{2} / \cancel{\binom{11}{4}}}{\cancel{\binom{4}{1}}\binom{7}{3} / \cancel{\binom{11}{4}}}$$

$$= \frac{12}{35}$$

3. Each day, you toss 10 coins. Call a day *lucky* if all 10 coins come out Heads. Let X be the number of lucky days among first $512 = 2^9$ days. Let Y the the number of lucky days among the next $256 = 2^8$ days (starting from the day 513).

(a) Identify the probability mass functions of X and Y. Compute EX and EY. Are X and Y independent?

X is Binomial $(2^9, \frac{1}{2^{10}})$ $\quad EX = \frac{1}{2}$

Y is Binomial $(2^8, \frac{1}{2^{10}})$ $\quad EY = \frac{1}{4}$

They are independent, as they count lucky days in disjoint subsets of days.

(b) Using the relevant approximation, estimate the probability $P(X \geq 3)$.

X is approx. Poisson $(\frac{1}{2})$

$$P(X \geq 3) \approx 1 - P(X=0) - P(X=1) - P(X=2)$$

$$= 1 - e^{-1/2} - \tfrac{1}{2} e^{-1/2} - \frac{(1/2)^2}{2} e^{-1/2}$$

$$= 1 - (1 + \tfrac{1}{2} + \tfrac{1}{8}) e^{-1/2} = \underline{\underline{1 - \tfrac{13}{8} e^{-1/2}}}$$

(c) Using the relevant approximation, estimate the probability $P(X \geq 3, Y = 0)$.

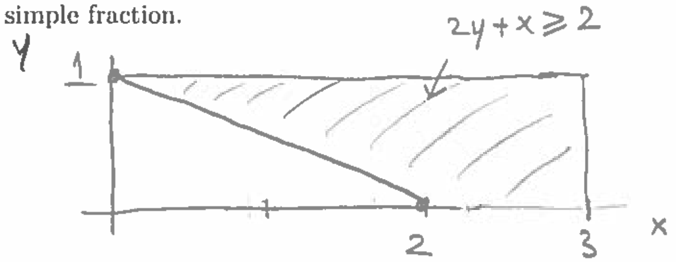
By independence, and as Y is approx. Poisson$(\frac{1}{4})$,

$$P(X \geq 3, Y=0) = P(X \geq 3) P(Y=0)$$

$$\approx \underline{\underline{\left(1 - \tfrac{13}{8} e^{-1/2}\right) \cdot e^{-1/4}}}$$

4. Select a random point (X, Y) in the rectangle $\{(x, y) : 0 \le x \le 3, 0 \le y \le 1\}$. Call the point *good* if $2Y + X \ge 2$.

(a) Compute the probability that the selected point is good. Draw a picture and give the result as a simple fraction.



P(the pt. is good)

$$= \frac{1}{3}(3-1) = \frac{2}{3}$$

(b) Now select, independently, 45,000 such random points. Using the relevant approximation, estimate the probability that at least 30,050 of them are good.

The no. X of good pts is Binomial$(45{,}000, \frac{2}{3})$

By Normal approx.:

$$P(X \ge 30{,}050) = P\left(\frac{X - 45{,}000 \cdot \frac{2}{3}}{\sqrt{45{,}000 \cdot \frac{2}{3} \cdot \frac{1}{3}}} \ge \frac{30{,}050 - \overbrace{45{,}000 \cdot \frac{2}{3}}^{30{,}000}}{\sqrt{\underbrace{45{,}000 \cdot \frac{2}{3} \cdot \frac{1}{3}}_{10{,}000}}} \right)$$

$$\approx P\left(Z \ge \frac{50}{100}\right) = P\left(Z \ge \frac{1}{2}\right)$$

$$= 1 - \Phi(0.5) \approx 1 - 0.69 = \underline{\underline{0{,}31}}$$