Math 16A, Winter 2016
Mar. 19, 2016.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the eight problems has equal worth. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 11 pages (including this one) with 8 problems. Read through the entire exam before beginning to work.

1	
2	
3	
4	
5	
6	
7	
8	
TOTAL	

1.

Compute the derivatives of the following two functions. *Do not simplify!* (Please be careful; you will receive little or no credit if you make a differentiation mistake, even a small one.)

(a) $y = \sqrt{x} \cdot (3 + 2\cos x)^4$

$$y' = \frac{1}{2} x^{-1/2} (3 + 2\cos x)^4$$
$$+ \sqrt{x} \cdot 4 (3 + 2\cos x)^3 (-2\sin x)$$

(b) $y = (x^3 + x)^{-8}$

$$y' = -8 (x^3 + x)^{-9} \cdot (3x^2 + 1)$$

2. Find the equation (in the slope-intercept form) of the tangent line to the curve

$$(x+2y)^4 + x^2y - x + y^3 = 0$$

at the point $(1, 0)$. (Please be careful; you will receive little or no credit if you make a differentiation mistake, even a small one.)

$$4(x+2y)^3(1+2y') + 2xy + x^2y'$$
$$-1 + 3y^2y' = 0$$

Plug in $x=1$, $y=0$:

$$4(1+2y') + y' - 1 = 0$$

$$9y' + 3 = 0$$

$$y' = -\frac{1}{3} \leftarrow \text{slope}$$

Line:

$$y - 0 = -\frac{1}{3}(x-1)$$

$$\underline{\underline{y = -\frac{1}{3}x + \frac{1}{3}}}$$

3. Compute the following limits.

(a) $\lim_{h\to 0} \frac{(1+h)^{-8}-1}{h}$

This is $f'(1)$ for $f(x) = x^{-8}$

The result is $f'(x) = -8x^{-9}$,

evaluated at $x=1$: $\underline{\underline{-8}}$.

(b) $\lim_{x\to 2} \frac{x^2-4}{\sqrt{x+2}-2}$ $\cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2}$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x+2-4}} \cdot (\sqrt{x+2}+2)$$

$= \underline{\underline{16}}$

4. Consider the function

$$f(x) = \begin{cases} ax^2 + b, & x < 1 \\ \frac{2}{x}, & x \geq 1 \end{cases}$$

$$f'(x) = \begin{cases} 2ax & x<1 \\ -\frac{2}{x^2} & x>1 \end{cases}$$

(a) Determine the numbers a and b so that $y = f(x)$ is differentiable for all x.

Cont. at $x=1$: $a+b=2$

Diff. at $x=1$: $2a=-2$

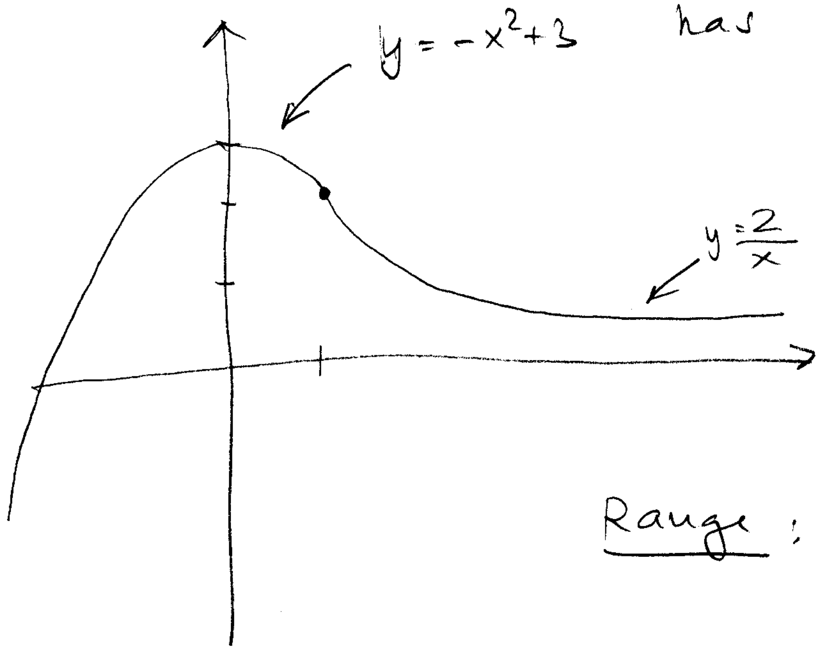
$\underline{a=-1, \quad b=3}$

$$f(x) = \begin{cases} -x^2+3 & x<1 \\ \frac{2}{x} & x \geq 1 \end{cases}$$

$$f'(x) = \begin{cases} -2x & x<1 \\ -\frac{2}{x^2} & x>1 \end{cases}$$

(b) Assume the values of a and b obtained in (a). Sketch the graph of the function f using the first derivative and determine its range.

$y=-x^2+3$ has max. at $x=0$



$y=\frac{2}{x}$ is decreasing on $[1,\infty)$

$\underline{\text{Range}}$: $\underline{(-\infty, 3]}$

5. Throughout this problem, the function $y = f(x)$ is given by $f(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$.

(a) Determine the domain of $y = f(x)$ and its intercepts. Compute also $\lim_{x\to 0+} f(x)$ and $\lim_{x\to\infty} f(x)$.

Domain: $x > 0$, No intercepts

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x}} = \infty,$$

$$\lim_{x\to 0+} \frac{\overset{\approx 1}{x+1}}{\underset{\text{small} > 0}{\sqrt{x}}} = +\infty$$

(b) Determine the intervals on which $y = f(x)$ is increasing and the intervals on which it is decreasing. Identify all local extrema.

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2}x^{-3/2}(x-1)$$

c.p. $x = 1$

	$(0,1)$	$(1,\infty)$
f'	$-$	$+$
f	$\searrow$	$\nearrow$

Local min at $(1,2)$

(c) Determine the intervals on which $y = f(x)$ is concave up and the intervals on which it is concave down. Identify all inflection points. (Use $4/\sqrt{3} \approx 2.3$.)

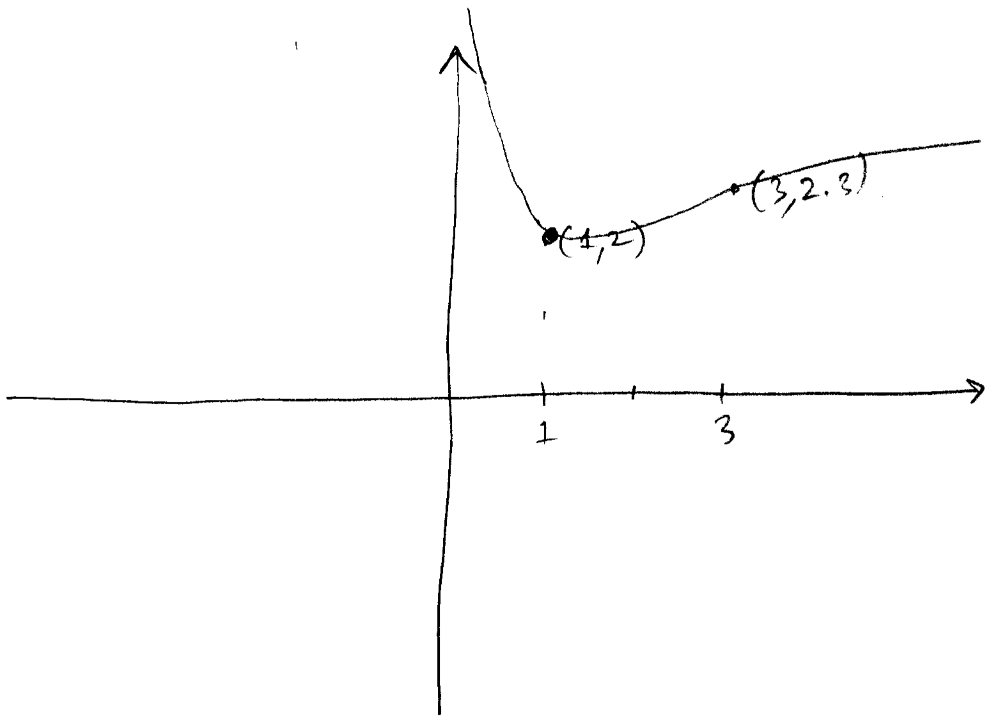
$$f''(x) = -\frac{1}{4}x^{-3/2} + \frac{3}{4}x^{-5/2}$$

$$= -\frac{1}{4}x^{-5/2}(x-3) \quad = 0 \text{ at } x = 3$$

	$(0,3)$	$(3,\infty)$
f''	$+$	$-$
f	conc. up $\smile$	conc. down $\frown$

$\left(3, \frac{4}{\sqrt{3}}\right) \approx (3, 2.3)$ infl. pt.

(d) (Still $f(x) = \dfrac{x+1}{\sqrt{x}}$.) Sketch the graph of $y = f(x)$. Label all points of importance on the graph.



(e) Determine the domain and range of the composite function $y = f(\sin x + 2)$.

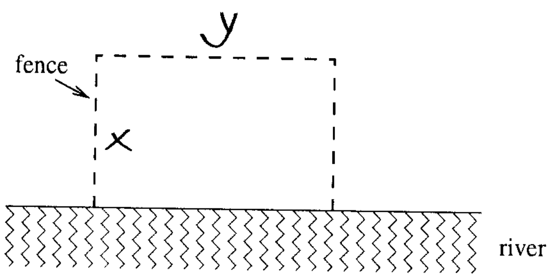
$\sin x + 2$ has values in $[1, 3]$

~~B~~ As f is defined on $[1,3]$, the domain are all x, and the range

$[2, 4/\sqrt{3}]$.

6. A farmer wants to fence off a rectangular field of area 8 square miles along the river, with no fence next to the river.

(a) What is the smallest length of fence the farmer can use?



$L = y + 2x$ $\qquad$ $xy = 8$ $\qquad$ $y = \frac{8}{x}$

$L = \frac{8}{x} + 2x$ $\qquad$ x in $(0, \infty)$.

$$\frac{dL}{dx} = -\frac{8}{x^2} + 2 = 2\,\frac{x^2 - 4}{x^2} = 0 \text{ when } x = 2.$$

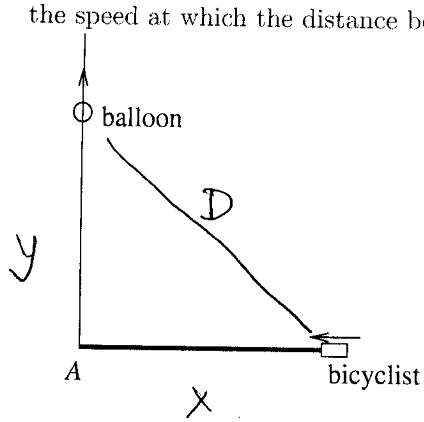
	$(0,2)$	$(2,\infty)$
$\frac{dL}{dx}$	$-$	$+$
	$\searrow$	$\nearrow$

Min. at $x = 2$, $y = 4$, $\underline{\underline{L = 8}}$ (miles)

(b) Assume now, additionally, that the length of the fence that goes parallel to the river must be at least 8 miles. What is now the smallest length of fence the farmer can use?

Then x is at most 1, and L decreases when x is in $(0, 1)$, so it achieves min. at $x = 1$, $y = 8$, $\underline{\underline{L = 10}}$.

7. A balloon is rising vertically above a straight trail, starting at the point A in the figure. A bicyclist is riding slowly on the trail *towards* A. At some point in time, the balloon is at 3 m above A, rising at the speed of 1 m/s, while the bicyclist is 4 m from A, moving at the speed of 3 m/s. Determine the speed at which the distance between the bicyclist and the balloon is changing at that instance.



$$D^2 = x^2 + y^2$$

$$\cancel{2} D \frac{dD}{dt} = \cancel{2} x \frac{dx}{dt} + \cancel{2} y \frac{dy}{dt}$$

When $x = 4$, $y = 3$, $D = \sqrt{16+9} = 5$,

and $\frac{dx}{dt} = -3$, $\frac{dy}{dt} = 1$, so

$$\frac{dD}{dt} = \frac{1}{5}\left(4(-3) + 3 \cdot 1\right) = -\frac{9}{5} \quad (m/s)$$

8. At price \$15 per car wash, the *Elephant* car wash expects 50 customers per day. At the current price of \$10, it has 100 customers per day. Elephant's fixed daily operating costs are \$200, and each car wash costs Elephant \$4. As usual, assume that the demand function is linear.

(a) Write down the demand function, that is, express the selling price p in terms of the number x of weekly customers. As usual, assume that the demand function is linear. Identify the proper interval for x.

x	p
100	10
50	15

$$p - 10 = \frac{10-15}{100-50}(x-100) = -\frac{5}{50}(x-100) = -\frac{1}{10}x + 10$$

$$p = -\frac{1}{10}x + 20 \qquad 0 \le x \le 200$$

(b) Express the *Elephant*'s weekly revenue R, cost C, and profit P as a function of x.

$$R = xp = -\frac{1}{10}x^2 + 20x$$

$$C = 4x + 200 \qquad P = R - C$$

$$P = -\frac{1}{10}x^2 + 16x - 200$$

(c) Which price should *Elephant* charge to maximize its profit? How many customers per week it would then have?

$$\frac{dP}{dx} = -\frac{1}{5}x + 16 = 0 \quad \text{when}$$

$$x = 5 \cdot 16 = 80$$

	$(0,80)$	$(80,200)$
$\frac{dP}{dx}$	+	−
P	↗	↘

Max at $\underline{\underline{x = 80}}$,

$\underline{\underline{p}} = -8 + 20$
$= \underline{\underline{12}}$

8. *Elephant car wash, continued.*
(d) Assume now that the *Elephant* needs to *add* to its daily cost (obtained in part (b)) a payment to its parent company. This additional daily payment depends on the number of customers x and amounts to $1600/(x+1)$ dollars. Write down the new cost function and find the number of customers that minimizes this cost. (This part of the problem is *only about cost*, so you can ignore revenue and profit.)

Now:
$$C = 4x + 200 + \frac{1600}{x+1}$$

$$\frac{dC}{dx} = 4 \cancel{x} - \frac{1600}{(x+1)^2}$$

$= 400$

$1 = 20$

$x = 19$

x	$(0, 19)$	$(19, 200)$
$\frac{dC}{dx}$	$-$	$+$
C	$\searrow$	$\nearrow$