Math 16C, Spring 1997.
Feb. 3, 1997.

MIDTERM EXAM 1

NAME(print):

NAME(sign): KEY

ID#:

Instructions: Each of the four problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 5 pages (including this one) with 4 problems. Read through the entire exam before beginning to work.

1	
2	
3	
4	
TOTAL	

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

1. Find the general solution to each of the following differential equations. You can leave the solutions in the implicit form.

(a) $yy' - 2xe^x = 0$

$$y\frac{dy}{dx} = 2xe^x$$

$$\int y\,dy = \int \underbrace{2x}_{u}\underbrace{e^x\,dx}_{dv}$$

$$\frac{y^2}{2} = 2xe^x - 2\int e^x dx$$

$$= 2xe^x - 2e^x + C$$

$$\boxed{\frac{y^2}{2} = 2xe^x - 2e^x + C}$$

(b) $xy' + y = x^2 + 1$ (Assume $x > 0$.)

$$y' + \frac{1}{x}y = x + \frac{1}{x}$$

$$P = \frac{1}{x} \qquad Q = x + \frac{1}{x}$$

$$u = e^{\int P(x)dx} = e^{\ln x} = x$$

$$\underline{y} = \frac{1}{u}\int (Qu)\,dx = \frac{1}{x}\int (x^2+1)\,dx$$

$$= \underline{\frac{1}{x}\left(\frac{x^3}{3} + x + C\right)}$$

$$\boxed{y = \frac{x^2}{3} + 1 + \frac{C}{x}}$$

2. Find the function $y = f(x)$ such that the point (1,5) lies on its graph and it satisfies the following differential equation:

$$x^2y' - 2xy = 12.$$

$$y' - \frac{2}{x}y = \frac{12}{x^2}$$

$$P = -\frac{2}{x} \qquad Q = \frac{12}{x^2}$$

$$u = e^{\int P dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$y = x^2 \int \frac{1}{x^2} \cdot \frac{12}{x^2}\, dx = x^2\left(\frac{12\,x^{-3}}{-3} + C\right)$$

$$= -4 \cdot \frac{1}{x} + Cx^2$$

$x=1,\ y=5$

$$5 = -4 + C \qquad C = 9$$

$$\boxed{y = -\frac{4}{x} + 9x^2}$$

3. A sphere is given by the equation

$$x^2 + y^2 + z^2 + 2x + 4y - 4z +2 = 0$$

(a) Does the point $(1, 1, 1)$ lie inside or outside the sphere?

$(x+1)^2 + (y+2)^2 + (z-2)^2 - 1 - 4 - 4 + 2 = 0$

$(x+1)^2 + (y+2)^2 + (z-2)^2 = 7$

Center: $(-1, -2, 2)$. Radius: $\sqrt{7}$

Distance d between center and $(1, 1, 1)$:

$d^2 = 2^2 + 3^2 + 1 = 14 > 7$

OUTSIDE

(b) For each of the three coordinate axes (the x-axis, the y-axis and the z-axis), determine whether the sphere intersects it.

x-axis $y = z = 0$ $(x+1)^2 + 4 + 4 = 7$ NO.

y-axis $x = z = 0$ $1 + (y+2)^2 + 4 = 7$ YES.

$(y+2)^2 = 2$ $y = -2 \pm \sqrt{2}$

z-axis $x = y = 0$ $(z-2)^2 = 2$ YES

$z = 2 \pm \sqrt{2}$

4. An 120 gallon container contains a mixture of 30% acid and 70% water. A mixture of 80% acid and 20% water is added at the rate of 3 gallons per minute, and the tank is drained at the same rate.

(a) Assuming that the liquid in the container is well-mixed, determine the time when the concentration of acid is 50%.

x = amount of acid

$$\frac{dx}{dt} = -3 \cdot \underbrace{\frac{x}{120}}_{\text{concentration of acid in the drained mixture}} + 3 \cdot \underbrace{0.8}_{\text{concentration of acid in the added mixture}}$$

$$\frac{dx}{dt} + \frac{1}{40}x = 2.4$$

$P = \frac{1}{40}$ $\quad u = e^{t/40}$ $\quad y = e^{-t/40}\int 2.4\, e^{+t/40}\, dx$

$$y = e^{-t/40} \cdot 2.4 \left(40 e^{t/40} + C\right)$$

$$= 2.4 \cdot 40 + C e^{-t/40} = 96 + C e^{-t/40}$$

$t = 0$, $y = 36$, $\quad C = -60$

$$y = 96 - 60 e^{-t/40}$$

$$60 = 96 - 60 e^{-t/40}$$

$$36 = 60 e^{-t/40}$$

$$\boxed{t = -40 \ln 0.6}$$

(b) Determine the limit of ~~concentration~~ the amount of acid in the container as time goes to infinity.

$\boxed{96}$ $\quad (= 80\%$ of $120)$