Math 16C, Winter 2020.
Feb. 5, 2020

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctors have been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. Find the general solution to each of the following differential equations. You may leave the solutions in the implicit form.

(a) $y' = \dfrac{x^3 + \sin x}{y^3}$

$$\frac{dy}{dx} = \frac{x^3 + \sin x}{y^3}$$

$$y^3 dy = (x^3 + \sin x)\,dx \qquad \int y^3 dy = \int (x^3 + \sin x)\,dx$$

$$\frac{y^4}{4} = \frac{x^4}{4} - \cos x + C$$

(b) $y' + 2xy = 4x$

$$P = 2x, \quad Q = 4x$$

$$u = e^{\int P dx} = e^{\int 2x\,dx} = e^{x^2}$$

$$y = \frac{1}{u}\int uQ\,dx = e^{-x^2} \cdot \int 4x\, e^{\boxed{x^2}} dx$$

$$x^2 = z, \quad 2x\,dx = dz$$

$$= e^{-x^2}\left[2\int e^z dz\right] = 2e^{-x^2}\left[e^z + C\right]$$

$$= 2e^{-x^2}\left[e^{x^2} + C\right] = 2 + Ce^{-x^2}$$

2. Find the function $y = f(x)$ such that the point $(1, 4)$ lies on its graph and it satisfies the following differential equation:

$$x^2 y' - xy = x - 2$$

$$y' - \frac{1}{x} y = \frac{1}{x} - \frac{2}{x^2} = x^{-1} - 2x^{-2}$$

$$P = -\frac{1}{x} \qquad Q = \frac{1}{x} - \frac{2}{x^2}$$

$$u = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\underline{\underline{y}} = \frac{1}{u} \int Q u \, dx = x \int (x^{-2} - 2x^{-3}) \, dx$$

$$= x \left[-x^{-1} + x^{-2} + C \right]$$

$$= \underline{\underline{-1 + \frac{1}{x} + Cx}} \quad \text{General solution,}$$

To determine C, plug in $x = 1$, $y = 4$:

$$4 = -1 + 1 + C, \quad \underline{C = 4}$$

<u>Answer:</u> $\underline{\underline{y = -1 + \frac{1}{x} + 4x}}$

3. A sphere which has a diameter with endpoints $(1,7,4)$ and $(3,1,2)$.
(a) Find the center and the radius of the sphere.

Center: $\left(\frac{1+3}{2}, \frac{7+1}{2}, \frac{4+2}{2}\right) = (2,4,3)$

Radius: distance between $(3,1,2)$ and $(2,4,3)$:

$$\sqrt{(3-2)^2+(1-4)^2+(2-3)^2} = \sqrt{1+9+1} = \sqrt{11}$$

(b) Does the point $(4,5,5)$ lie inside or outside the sphere?

The square of the distance between $(4,5,5)$ and $(2,4,3)$ is

$$(4-2)^2+(5-4)^2+(5-3)^2 = 4+1+4 = 9 < 11$$

<u>Inside</u>.

4. A 100 gallon container initially contains 5 gallons of alcohol an 95 gallons of water. A mixture of 25% alcohol and 75% water is added at the rate of 2 gallons per minute, and the tank is drained at the same rate.

(a) Assuming that the liquid in the container is well-mixed, determine the amount of alcohol in the container after 25 minutes.

y = amount of alcohol in the container
t = time in minutes

$$\frac{dy}{dt} = -2\,\frac{y}{100} + 2\cdot\frac{1}{4} = -\frac{y}{50} + \frac{1}{2}$$

$$\frac{dy}{dt} + \frac{y}{50} = \frac{1}{2} \qquad P = \frac{1}{50},\quad Q = \frac{1}{2}$$

$$u = e^{\int P\,dt} = e^{\frac{1}{50}t}, \quad y = \frac{1}{u}\int uQ\,dt = e^{-\frac{1}{50}t}\int e^{\frac{1}{50}t}\cdot\frac{1}{2}\,dt$$

$$\text{so, } y = e^{-\frac{1}{50}t}\cdot\left[\frac{1}{2}\cdot 50\,e^{\frac{1}{50}t} + C\right]$$

$$= 25 + Ce^{-\frac{1}{50}t}$$

When $t=0$, $y=5$, so $5 = 25 + C$, $C = -20$

$y = 25 - 20e^{-\frac{1}{50}t}$, When $t=25$, $\underline{y = 25 - 20e^{-1/2}}$

(b) Determine the limit of the amount of alcohol in the container as time goes to infinity.

As $t \to \infty$, $y \to 25$.