Math 17C, Spring 2011.

Linear Systems, Complex Eigenvalues

We have a two-dimensional system

$$\frac{d\vec{x}}{dt} = A\vec{x},$$

where A has complex eigenvalues $\lambda_{1,2} = \alpha \pm i\beta$. Assume that $\vec{u} + i\vec{v}$ is an eigenvector for $\lambda_1 = \alpha + i\beta$. In components,

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

We will now explain how we get the general solution from this information, but if you are interested only in the recipe, here it is. The general solution is given by

$$c_1 e^{\alpha t} \begin{bmatrix} u_1 \cos(\beta t) - v_1 \sin(\beta t) \\ u_2 \cos(\beta t) - v_2 \sin(\beta t) \end{bmatrix} + c_2 e^{\alpha t} \begin{bmatrix} u_1 \sin(\beta t) + v_1 \cos(\beta t) \\ u_2 \sin(\beta t) + v_2 \cos(\beta t) \end{bmatrix},$$

where c_1 and c_2 are arbitrary constants.

To justify (and possibly help to remember) this formula, write a solution in the form

$$e^{\lambda_1 t}(\vec{u} + i\vec{v}) = e^{(\alpha + i\beta)t}(\vec{u} + i\vec{v}) = e^{\alpha t} \cdot e^{i\beta t}(\vec{u} + i\vec{v}).$$

The main insight is the interpretation of $e^{i\beta t}$. If one writes this expression as an infinite series (something we do not learn in this course), one gets the *Euler formula*:

$$e^{i\beta t} = \cos(\beta t) + i \sin(\beta t).$$

Then,

$$e^{\lambda_1 t}(\vec{u} + i\vec{v}) = e^{\alpha t} \cdot (\cos(\beta t) + i \sin(\beta t))(\vec{u} + i\vec{v}) = e^{\alpha t} \cdot (\cos(\beta t) + i \sin(\beta t)) \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right).$$

Now, if one multiplies the complex numbers out, one gets

$$e^{\alpha t} \begin{bmatrix} u_1 \cos(\beta t) - v_1 \sin(\beta t) + i\,(u_1 \sin(\beta t) + v_1 \cos(\beta t)) \\ u_2 \cos(\beta t) - v_2 \sin(\beta t) + i\,(u_2 \sin(\beta t) + v_2 \cos(\beta t)) \end{bmatrix}.$$

The real and imaginary parts of this expression are two linearly independent solutions whose linear combination is the general solution in the recipe.

Example. Let

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}$$

Find the general solution and solve the initial value problem with $x_1(0) = 3$, $x_2(0) = 1$.

As $\det(A - \lambda I) = \lambda^2 - 6\lambda + 10$, we get $\lambda = 3 \pm i$. Let us find the eigenvector for $\lambda_1 = 3 + i$. We will solve

$$(A - \lambda_1 I)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.$$

This leads to two equations for z_1 and z_2, but one of the equations is superfluous. The first equation says

$$(-1 - i)z_1 + z_2 = 0.$$

We may, for example, choose $z_1 = 1$ and then $z_2 = 1 + i$. Thus

$$\vec{u} + i\vec{v} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The general solution then is

$$c_1 e^{3t}\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} \sin t \\ \sin t + \cos t \end{bmatrix}.$$

When $t = 0$, this gives

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix}.$$

We get

$$\begin{bmatrix} c_1 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

so $c_1 = 3$, $c_2 = 1 - c_1 = -2$. Finally,

$$\vec{x}(t) = 3e^{3t}\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} - 2e^{3t}\begin{bmatrix} \sin t \\ \sin t + \cos t \end{bmatrix} = \begin{bmatrix} e^{3t}(3\cos t - 2\sin t) \\ e^{3t}(\cos t - 5\sin t) \end{bmatrix}.$$

Discussion and homework problems. Find the solution in each of the following cases.

(a) $A = \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}, \quad x_1(0) = 1, x_2(0) = 1.$

(b) $A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad x_1(0) = 1, x_2(0) = 1.$

(c) $A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}, \quad x_1(0) = 1, x_2(0) = -1.$

(d) $A = \begin{bmatrix} -4 & 5 \\ -4 & 4 \end{bmatrix}, \quad x_1(0) = 1, x_2(0) = 2.$