Math 189, Winter 2008.

Final Exam

Give clearly written solutions in the space provided. If you need more space, attach clearly labeled extra pages. You do not need to type the solutions.

You should work on *every* problem. First six problems are easier than the rest. None of them are impossibly hard, or very long.

The exam in due in lecture on Friday, March 14.

All solutions you give will be graded by math competition standards: you will receive credit only for *correct* arguments that *clearly* lead to a solution.

NAME: ____

1. Call a natural number n *green* if $\pm 1 \pm 2 \cdots \pm n = 0$ for some combination of the signs. Find a simple characterization of green numbers. In particular, determine which of $2008, 2009, 2010, 2011$ are green.

2. For which numbers $a > 0$ is $a^x \geq x$ for every $x \in \mathbb{R}$?

3. Fix a positive integer a. Show that the Fibonacci sequence f_n, given by $f_0 = 0,\ f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$, contains infinitely many multiples of a.

4. Given an integer n, let $f(n)$ denote the number of ordered pairs of nonempty, disjoint subsets of $\{1, 2, \ldots, n\}$. Find a simple formula for $f(n)$.

5. Let A be the set of positive integers with no 9 among their decimal digits. Determine whether

$$\sum_{n \in A} \frac{1}{n}$$

converges.

6. Start with a vector a_1 with n entries, all positive integers. Given the vector a_n, the i'th entry of the vector a_{n+1} is the number of occurencies, in a_n, of the i'th entry of a_n. Show that this recursively defined sequence changes only finitely many times.

Here is an example with exactly two changes:

$$(1, 3, 3, 1, 1, 5, 5), (3, 2, 2, 3, 3, 2, 2), (3, 4, 4, 3, 3, 4, 4).$$

7. Find the precise set of numbers γ for which the double sum below converges, i.e.,

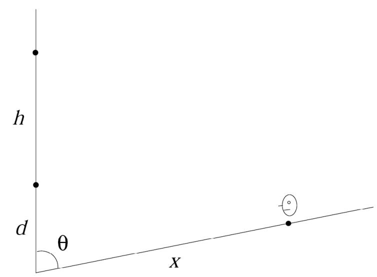
$$\sum_{m,n=1}^{\infty} \frac{1}{(m+\sqrt{n})^{\gamma}} < \infty.$$

8. You are in posession of n pairs of socks (hence a total of $2n$ socks) ranging in shades of grey, labeled from 1 (white) to n (black). Take the socks blindly from a drawer and pair them at random. What is the probability that they are paired so that the colors of any pair differ by at most 1? You have to give an explicit formula, which may include factorials.

9. Each element of an 11×11 matrix $[a_{ij}]_{i,j=1}^{11}$ is either 1, 2, or 3. Prove that there exists a rectangle with four corner vertices equal, that is, $1 \leq i_1 \leq i_2 \leq 11$ and $1 \leq j_1 \leq j_2 \leq 11$ so that $a_{i_1 j_1} = a_{i_1 j_2} = a_{i_2 j_1} = a_{i_2 j_2}$.

10. Find all integer solutions to $x^2 + 615 = 2^n$.

11. A movie theater has screen of height h mounted on a vertical wall so that the bottom of the screen is at height d. The floor with the seats is at the angle θ with the screen wall. Measure the distance from the screen wall along the floor with seats (as x in the picture below). What distance from the screen is the best, that is, it gives you the largest angle of view of the screen?



12. Start with $a_1 = 1$, and let a_{n+1} be the smallest positive integer not of the form $a_i + i$, $1 \leq i \leq n$. Thus the sequence begins with $1, 3, 4, 6, 8, 9, 11, 12, 14, \ldots$ Show that $a_n = \lfloor nx \rfloor$ for a number x.