Problem Set 1: Introductory Problems

Here is a list of 42 problems. Most (if not all) of these were asked at interviews at high tech companies, despite the fact that they are long-known classics. They have been chosen for this set because they have a relatively quick ("Aha!") solutions, and there is no guiding principle to solve them. *Important note*: One of the mistake that applicants often make is that they ask the interviewer to "please precisely define" all the terms in the puzzle. It is *your responsibility* to make reasonable assumptions so that the puzzle and its solution are meaningful. However, you can assume that there are no trick questions with completely trivial answers.

Another note: The most common formulations of the puzzles are given, without any attempt to make them more politically correct. Especially Wall Street interview questions seem to have a taste for the macabre (see for example the Russian roulette puzzle below).

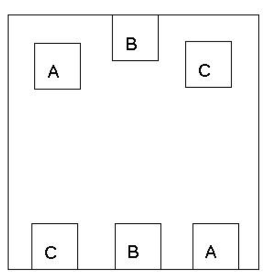
1. You have two jars, 50 red an 50 blue marbles. You can distribute the marbles in the two jars as you wish. Then one of the two jars is chosen at random, and finally a marble is picked at random from the chosen jar. How do you maximize the probability that a red marble is picked?

2. After you express an opinion, somebody says: "I cannot fail to disagree with you less." Is he on your side or not?

3. How many times a day do the hour and minute hands on an analog clock overlap?

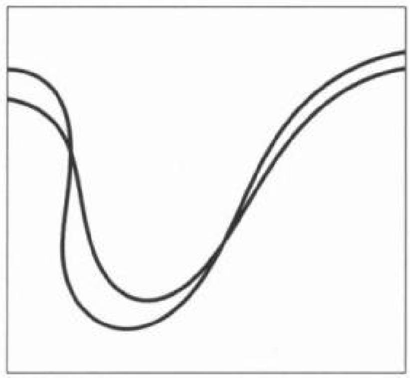
4. You find yourself in a shooting match with two other desperadoes, Bob and Alice. You shoot with 1/3 accuracy, while Bob and Alice shoot with 2/3 and 100% accuracy, respectively. There is one shot per person per round, and each round goes from worst to best shooter, until only one remains standing. What is your strategy?

5. Connect the two A-squares, the two B-squares and the two C-squares with continuous nonintersecting curves inside the large square.



6. Sherlock Holmes and Dr. John Watson come across the bicycle tracks in the mud as shown in the picture below. Watson observes that the tires do not have directional treads, so there is

no way to tell whether the bicycle went to the left or to the right. Holmes, however, disagrees. Why?



Note. The Adventure of the Priory School, one of the 56 Sherlock Holmes short stories written by Arthur Conan Doyle, features an *inadequately supported* solution to this problem. If interested, read the original story or the article on Wikipedia.

7. A tennis tournament proceeds by single elimination. This means that a number n of players enter the contest, they are paired as much as possible in the first round. That is, if n is even, $n/2$ matches are played, while if n is odd $(n-1)/2$ matches are played and one player receives a bye. After the games are played, all losers are out, while the winners and the bye player (if any) continue to the second round, which proceeds by the same rules. This continues until the final match, which determines a single winner. Assume 2007 players enter a tournament. How many games are played?

8. Two locomotives are heading toward each other on a straight track, starting at distance 20 miles and traveling at 10mph. At the instant they begin, a fly takes off from the front of one train at the speed of 15 mph, and flies straight towards the second train. Upon encounter, it immediately reverses direction. So it continues back and forth until it is crushed in the collision of two locomotives. What is the total distance traveled by the fly?

9. An explorer walks one mile due South, the one mile due East, then one mile due North. He finds himself back where he started. Where could he have started?

10. You have a rectangular chocolate bar, which is divided into $m \times n$ squares in the usual fashion. You want to break this bar into its constituent squares. Each step consists of picking one of the leftover pieces and breaking it into two pieces along one of the (vertical or horizontal) lines. (For example, you have $m + n - 2$ choices for the first step.) How many steps do you need?

11. A die has 6 faces, each of which has a different number among $1, \ldots, 6$ on it. How many different dice are there?

12. Three coworkers want to know their average salary, but do not want to disclose their individual salaries. Can they do this? Assume they have no gadgets and can only talk to each other (but two can talk without being heard by the third).

13. You are a broker; your job is to accommodate your client's wishes without placing any of your personal capital at risk. Your client wishes to place a \$1,000 bet on the outcome of the World Series, which is a baseball contest decided in favor of whichever of two teams first wins 4 games. The client deposits his \$1,000 with you in advance of the series. At the end of the series he must receive from you either \$2,000 if his team wins, or nothing if his team loses. No market exists for bets on the entire world series. However, you can place even-odds bets, in any amounts, on each game individually. (You should know this: an even-odds bet in the amount x means that you win x if "your" outcome happens and lose x otherwise.) What is your strategy for placing bets on the individual games in order to achieve the cumulative result demanded by your client?

14. For each square of a standard 8×8 chessboard, you either put a coin on it or leave it empty. You also have to guarantee that each row and column will contain an odd number of coins. How many such configurations of coins are there?

15. On a table there is a row of 50 coins, of various denominations. Alice starts by taking a coin from one end, then Bob takes a coin from one of remaining ends, etc., until Bob takes the last coin. Is there a strategy that lets Alice get at least as much money as Bob?

16. A coin collector has received n (say, 6) dirty coins. The tray in his coin washing machine only has space for k (say, 4) coins. In addition, each coin has to go in the machine twice, because it needs to be washed on both sides. A single washing takes an hour (independently on the number of coins on the tray). What's the minimal time in which all coins can be washed?

17. "Let's play a game of Russian roulette. You are tied to your chair and can't get up. Here's a gun. Here's the barrel of the gun, six chambers, all empty. Now watch me as I put *two* bullets in the gun, in *two adjacent chambers*. I close the barrel and spin it. I put a gun to your head and pull the trigger. Click. Lucky you! Now I'm going to pull the trigger one more time. Which would you prefer, that I spin the barrel first, or that I just pull the trigger?"

18. Your father owns a rectangular field, from which the city has appropriated a smaller rectangular patch. He wants to split the remainder between you and your brother so that each of you two gets equal area. How does he do this?

19. There are ants at three corners of a triangle. Each ant starts moving on an edge towards another vertex, chosen at random. What is the probability that no ants collide?

20. Four equally fast dogs are each at corner of a large square. Each of them begins chasing the dog clockwise from it. They all continuously adjust their direction so that they are always

heading straight toward their clockwise neighbor. How long does it take for them to catch each other and where does this happen?

21. You have three picnic baskets filled with fruit. One has apples, one has oranges, and the third has a mixture of the two. Each basket is clearly labeled, but each label is wrong. You are permitted to select a basket, then you are given a piece of fruit from it. Can you determine what is in each basket?

22. You have b boxes and n dollar bills. Distribute the money into boxes, then seal them. Thereafter, you have to be able to use the boxes to exactly pay any whole amount between 0 and n dollars. What are restrictions on b and n?

23. You have two lengths of fuse. Each will burn for exactly one hour. But the fuses are not identical and will not burn at a constant rate. How can you measure 45 minutes with the two fuses, and a lighter?

24. You are in a boat in the exact center of a circular lake. There is a goblin on the lake shore who wants to kill you. You can run faster than the goblin, but he runs 4 times as fast as you can row. Can you escape the goblin?

25. One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes the assigned seat if it is unoccupied, and one of unoccupied seats at random otherwise. What is the probability that the last person to board gets to sit in his assigned seat?

26. You die and the devil says he'll let you go to heaven if you beat him in a series of two games. On a table, there is a (standard 8×8) chessboard, and beside it is a pile of quarters. He says: "We'll take turns putting quarters down, on the squares of the board. I can put the coin on any empty square, you must try to put a coin on one of the four squares adjacent to the one I just occupied. The first one of us who can't put a quarter down loses. Of course I also start." How do you play?

27. Eager for revenge, the devil says, "OK, now for the second game. We lay down quarters as before, but now neither of us can put a quarter onto a square which is either occupied or adjacent to any of the occupied ones, otherwise there are no restrictions. Again, I start." How do you play this time?

28. You are in a room with 9 other job applicants, when suddenly security guards with submachine guns come in. The interviewer says: "The interview has reached the decisive stage. We will put you in a straight line in a random order. Each of you will receive either a blue or a red hat. You will only be able to see hats in front of you; if any of you tries to look at his or her own hat or turns around you will be all shot. One by one, starting from the last person in

the line, you will be asked for the color of your hat. Each of you who gives an incorrect answer will be shot immediately, while the survivors will be given job offers. I will now leave you a few minutes to decide about your strategy — you may coordinate among yourselves."

29. A person dies, and arrives at gates to heaven. There are three doors: one of them leads to heaven, another one leads to a 1-day stay in hell, and then back to the gate, and the other leads to a 2-day stay in hell, and then back to the gate. Every time the person is back at the gate, the three doors are reshuffled. How long, on the average, will it take the person to reach heaven?

Next three are famous "crossing" puzzles.

30. Three cannibals and three anthropologists have to cross a river. The boat they have is only big enough for two people. If at any point in time there are more cannibals on one side of the river (or on the side of the river plus in the boat that lands there) than anthropologists, the cannibals will eat them. What plan can the anthropologists devise for crossing the river so they don't get eaten? Of course, the boat can't cross the river by itself.

31. Four people need to cross a rickety rope bridge at night. Unfortunately, they only have one flashlight and it only has enough light left for seventeen minutes. With many planks missing, the bridge is too dangerous to cross without a flashlight, and it is only strong enough to support two people at any given time. Each of them walks at a different speed: Adam can cross the bridge in one minute; Larry in two minutes; Edge takes five minutes; and the slowest, Bono, needs ten minutes. (The U2 angle is a standard modern touch, although the puzzle is probably more that a hundred years old.) How do they make it across in seventeen minutes?

32. (*) A dysfunctional family has to cross the river. On one side of the river are: mom and two daughters, dad and two sons, the maid and the dog. There is a boat only big enough to hold two people (counting the dog as a person). Here are the difficulties. If the dog is left with anyone and the maid isn't there to control him, he'll bite. The dad can't be left with any of the daughters when the mom isn't there. Likewise, the mom can't be left with either of the sons when the dad isn't there. Finally, only an adult can operate the boat.

Next three are famous "weighing" puzzles. A *balance scale* can compare any two masses A and B and decide whether $A < B$, $A > B$ or $A = B$. An *analytic scale* can measure any given mass.

33. Assume that you have a balance scale. In your possession are n gold coins, all the same in appearance. However, $n - 1$ of them are genuine and have the same weight, while one is a fake and weights a little less. What is the minimal number of weighings which identifies the fake coin? In particular, solve the problem for $n = 120$.

34. Now you have an analytical scale, and n types of coins, all in infinite supply. Coins of the same type are identical, and any coin weighs an integer number of grams. How many weighings do you need to determine all the weights?

35. (*) Back to balance scale. Now you have 12 coins, one of which *may* be a fake. A fake coin has a different weight than a genuine coin, but you don't know whether it is lighter or heavier. How many weighings do you need in order to determine the identity of the fake coin, if any, and whether it is lighter or heavier? Then find a good solution for $c = (3^k - 3)/2$ coins, one of which is a fake.

36. (*) In the old fashioned dry goods stores, the weight was measured by a balance scale and a collection of 8 standard brass weights, of the following weights, in ounces: 1, 2, 2, 5, 10, 10, 20, 50. It is easy to see that any integer weight between 1 and 100 can be measured by these weights.

(a) What is the minimal number of brass weights one needs in order to measure any integer weight between 1 and 100 if one is allowed to put weights only on one side of the scale?

(b) Solve the same problem if one is allowed to put weights on both sides of the scale.

Solve both problems if 100 is substituted by a general n.

37. A tunnel underneath a large mountain range serves as a conduit for n identical wires, so you see n wire-ends at each end of the tunnel. Your job is to label each end by a number 1 to n so that each wire has the same label at its two ends. You may join together arbitrary groups of wires at either end; they will then conduct electricity through the join. Then you cross the mountain and use electricity to check which wires are joined. Then you make or unmake connections as desired, and cross the mountain to the other side, etc. How can you accomplish your task with the smallest number of mountain crossings?

38. (*) You have two identical crystal orbs and you are standing in front of a 100 story building. You want to know what is the highest floor (if any) such an orb can be dropped from and still survive the fall. What is the smallest number of drops which is guaranteed to give you this information? (You are allowed to break one or both orbs in your quest. Note that the answer is trivially 100 if you have just one orb.) Then solve Problem 14 for $3, 4, \ldots$ orbs.

39. (*) A clock has identical hour and minute hand. How many times a day are you *unable* to tell the time with such a clock (by a single glance, without observing the motion)? What if the clock also has the identical second hand?

40. Assume that you have a set S of k lattice points $(a_1, b_1), \ldots, (a_k, b_k) \in \mathbb{Z}^2$ in the plane. Those correspond to locations of stores in a city in which streets form a perfect square grid. You want to open the bakery, and locate it at a lattice point so that the sum of the distances to these stores is minimal, as this would minimize transportation costs. (Note that you can only move on the lines of the grid, so the distance is "taxicab distance," e.g., the distance between (1,2) and (4,6) is $3 + 4 = 7$.) Find the location(s) which achieve the minimal distance. For example, solve the problem with lattice points $(-1, 4)$, $(3, 1)$, $(-1, 1)$ and $(-4, 0)$.

41. (*) What is the number of ordered divisions of an integer n into positive summands? For instance, 4 can be divided in eight ways: $4 = 3 + 1 = 1 + 3 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 = 1 + 1 + 1 + 1$.

42. Assume that you have a penny and another coin of exactly three times larger diameter. Put the two coins next to each other on a flat surface and proceed to roll (withot slipping) the penny around the perimeter of the larger coin. What is the total angle by which the Lincoln's head will have turned when the penny reaches its starting position?

This famous problem appeared on a SAT exam in 1982, with the following choices, in multiples of 2π, for the answer: $3/2$, 3, 6, $9/2$, 9. None of these is correct! For a completely convincing argument, you should replace the larger coin with an arbitary planar convex set S of perimer p.