Problem Set 2: Logic and Information

The theme of this set is mathematical logic. To solve these problems, you will need to use such devices as the *law of excluded middle* (every statement is either true or untrue), *proof by contradiction* (when you try to prove something, you may assume its converse, in addition to stated assumptions) and *mathematical induction* (you can prove that natural numbers have property $\mathbf{P}$ by checking that 1 has this property, and that if n has property $\mathbf{P}$, then $n+1$ must have it as well).

1. A politician comes to a press conference, reads the following list of statements, then leaves.

"1. I have never taken a bribe.

2. Exactly two statements on this list are false.

3. Exactly three statements on this list are false."

Assuming every statement is either correct or incorrect, determine which is which.

Devise a similar list of 10 statements with similar unique solution.

2. A certain island contains two tribes: Liars (every statement they utter is false) and Truthtellers (every statement they utter is true).

The classic puzzle finds you traveling on the island, and arriving at a fork in the road. The two roads are labeled A and B, one of which will lead you to your destination. There is a native coming down the road, and you can ask him a single yes–no question which will determine which road you should take. What do you ask him?

Recent immigration has created a third tribe, Normals (their statements can be true or false). A crime is committed and three suspects A, B, and C are brought in for questioning. It is known that exactly one of them committed the crime.

(a) It becomes known that one of the three is Normal, one Liar, one Truthteller. Determine which is which on the basis of the following three statements they give:

A: I am a Normal.

B: The above A's statement is correct.

C: I am not a Normal.

(b) In a different situation, further evidence surfaces that a Truthteller is the criminal, and that there is exactly one Truthteller among the three suspects. This time the statements are:

A: I am innocent.

B: The above A's statement is correct.

C: B is not a Normal.

Who is the criminal?

(If you like such puzzles, pick up a book by Raymond Smullyan, such as *What Is the Name of This Book?: The Riddle of Dracula and Other Logical Puzzles*.)

3. This is the famous *unexpected hanging paradox*. A prisoner is sentenced to be hanged. "The hanging will take place at noon," adds the judge, "on one of the seven days, Monday through Sunday, of next week. But you will not know which day it is until you are so informed on the day of the hanging." This is how the prisoner reasons: "They can't hang me on Sunday, since in this case I will be able to predict this on Saturday afternoon. If they hang me on Saturday, I'll be able to predict this on Friday afternoon, so Saturday is ruled out too. So is Friday, by the same logic. So are Thursday, Wednesday, Tuesday and, finally, Monday. They can't hang me at all!" Yet the execution squad arrives on Thursday and catches the prisoner completely by surprise, vindicating the judge's word. Explain.

This particular story has generated hundreds of scientific papers, mostly in the field of philosophy. Read a review in *American Mathematical Monthly* 105 (1998), 41–51, if interested.

4. A Zen master devises the following test as a final exam for his 50 disciples. In the evening, before they retire to their cells, he tells them each of their foreheads is painted either black or white during the night. During the day's meditation, they will be able to see the others' foreheads, but not their own. There are no reflecting surfaces in the monastery, and the disciples have to observe a vow of silence during the exam. If a disciple is able to logically deduce the color of his forehead during a particular day, he passes the exam and can leave the monastery that night.

The master decides to paint all foreheads white. Next morning, during the prayer, the master also divulges the following apparently useless piece of information: "There is at least one forehead painted white." Or is it really useless? What if only one forehead were painted white? Or only two? How long will the exam go on?

Next year, the master has just two disciples, and as a final exam it tells one of them (in secret) "your number is 13," and the other "your number is 14." This time, if a disciple is able to deduce the other's number during a particular day, he passes the exam and can leave the monastery that night. As before, there's vow of silence. Next morning, during the prayer, the master says: "I've given you two consecutive integers, each at least 1." How long will the exam go on this time?

5. (*) The Zen master has n disciples, who are painted on their foreheads nonnegative numbers $a_1, \ldots, a_n$. The master also publicly announces the numbers $s_1 < \cdots < s_k$, $k \leq n$, and tells the disciples that one of the s's is the true sum of their numbers. For simplicity, you can assume that the a's are integers, although that is not essential. Whenever the first disciple knows his number (or, equivalently, the true sum), the exam is over. What will happen? What may happen if $k > n$?

6. A band of n pirates has just robbed a ship on an open sea. The loot consists of 10 valuable gold plates. The pirates are ranked $P1, P2, \ldots, Pn$ in increasing rank. This is how they divide the spoils (the individual plates are indivisible). The highest ranked pirate proposes a division, which is then voted on by all n of them. If at least $n/2$ approve, the division is accepted and the story ends. Otherwise, Pn is thrown overboard, and the procedure is repeated with $n-1$ pirates. This is how each pirate ranks his preferences, in decreasing order: getting one or more

plates (the more the better), seeing the spectacle of somebody being thrown overboard, getting nothing, being thrown overboard.

If $n = 2$, $P2$ clearly takes all 10 plates, as his vote alone assures the plan is approved. If $n = 3$, $P3$ has to give $P1$ a plate, hence assuring his vote ($P1$ knows he gets nothing if $P3$ is thrown overboard), so the split is (1,0,9) in this case. Determine the split for *every* $n \geq 4$.

7. (*) This is the famous *prisoners and the light bulb* puzzle. The version here is an adaptation from the *Car Talk* radio show, but this is another popular job interview problem.

A warden meets with n new prisoners when they arrive. He tells them:

> "You may meet today and plan a strategy. But after today, you will be in isolated cells and will have no communication with one another. In the prison is a switch room, which contains a light switch, which can turn the single light bulb in the room on or off. I am not telling you its present position. After today, from time to time whenever I feel so inclined, I will select one prisoner at random and escort him to the switch room. This prisoner may, but is not obligated to, reverse the position of the switch. Then he'll be led back to his cell. No one else will enter the switch room until I lead the next prisoner there, and he'll be instructed to do the same thing. I'm going to choose prisoners at random. However, this is not a probability problem — I'm only assuring you that, given enough time, every one of you will eventually visit the switch room arbitrarily many times. At any time anyone of you may declare to me, 'We have all visited the switch room.' If it is true, then you will all be set free. If it is false, and somebody has not yet visited the switch room, you will be fed to the alligators."

What is the strategy the prisoners devise?

8. Indiana Jones arrives at the ancient temple of the Sphinx. He hears a disembodied voice say: "You see in from of you 4 vases, each of which either contains a key or a deadly poisonous snake. The keys, if any, in the vases are needed for the admission into the temple." "How can I figure out which vases contain the keys," nervously asks Indiana, with bad guys at his heels and his well-known aversion to snakes. "You can ask me any number of yes–no questions, but I am permitted to lie to you at most once," answers the voice. Help Indiana get into the temple with the fewest number of questions. Then generalize to an arbitrary number n of vases.

9. Somebody has picked n integers $0 \leq a_1 < a_2 < \cdots < a_n$. These are unknown to you. The information you get are all $\binom{n}{2}$ sums of these numbers (in, say, increasing order, so you are *not* told which two numbers produce which sum). Can you determine the unknown numbers?

This problem demonstrates the value of selecting the proper tool. It is a very difficult problem to solve by brute force, so try to find the "right sledgehammer."

10. Another famous problem which has a nice solution when interpreted properly is the following. You have n lamps and each lamp has a switch. There also are several connections between pairs

of lamps. These connections are completely known to you, work the same in both directions, and have the following effect. If you flip a swich on a lamp, you change its state (from on to off or vice versa) and also the state of all lamps connected to it. Initially, all lamps are turned off. Prove that, regardless of the connectivity structure, you can find a sequence of flips that turns all the lights on.

Note that it is important that all the lights are initially off. Several versions of this puzzle exist (and were even sold) as a game, in which you are given a connectivity structure and a configuration of lamp states (some on and some off), and you need to figure out how to turn all of them on.