

Problem Set 3: Pigeonhole Principle

The problems in this set are related to the so-called *pigeonhole principle* or, if we wish to sound more sophisticated, the *Dirichlet exclusion principle* (in honor of the German mathematician J. P. G. Lejeune Dirichlet, 1804-1859, who called it *Schubfachprinzip* and first used it explicitly in number theory). The basic principle is *very* simple: *If $n + 1$ objects are placed in n boxes, at least one box has more than one object*, so it is somewhat surprising how varied and non-obvious its applications are. Most applications (as well as the principle itself) could be described as bookkeeping — categorize the possibilities in until you get what you look for. Some problems below can be solved by this general strategy rather than by a literal application of the pigeonhole principle.

1. There are n persons in a room, some of which are acquainted, some not. Assume that nobody is acquainted by himself or herself, but that the relation is symmetric (A knows B means B also knows A). Show that there are two people with the same number of acquaintances.

2. Here's a similar problem by M. Gardner: "My wife and I recently attended a party with four other married couples. At the beginning, various handshakes took place. No one shook hands with himself or herself, or with his or her spouse, or with somebody more than once. After all the handshakes were over, I asked each person, including my wife (a total of nine questions to nine people), how many hands he or she had shaken. Each gave a different answer. How many hands did my wife shake?"

3. Fix an integer $n \geq 1$. (a) Show that there exists a number divisible by n whose decimal digits are only 0 and 1. (b) Show that there exists a number divisible by 2^n whose decimal digits are only 1 and 2.

4. (*) A *polyhedron* is a tri-dimensional body bounded by polygons. These polygons are called *faces* of the polyhedron, they meet at *edges*, which in turn meet at *vertices*. Note that we are in general not assuming that a polyhedron is convex; in fact it can be arbitrarily twisted and even have holes in it.

Show that every polyhedron has two faces bounded by the same number of edges; in fact there are at least two disjoint pairs of faces with the same number of boundary edges.

5. The plane lattice consists of points with integer coordinates. Choose five points on the plane lattice. You can always choose two points out of this five, so that the segment connecting them contains another lattice point. Prove this.

6. Forty-one rooks are placed on a 10×10 chessboard. Show that there are five rooks that do not attack each other — that is, no two of these five share a row or a column.

7. Take 101 points in the unit square. Show that three of them are vertices of a triangle with area at most 0.01.

8. A region A in \mathbb{R}^3 has volume n . Show that some translate must cover n points in \mathbb{Z}^3 . (If you want to avoid technicalities, you can assume that A is union of finitely many cubes.)
9. Pick any n distinct points in the plane, then draw all the line segments between different pairs (all $\binom{n}{2}$ of them). Show that two of these segments emanating from the same point are at an angle of at most π/n . (Note: the angle 0 counts.)
10. Assume that α and β are positive irrationals. Let $\lfloor x \rfloor$ be the integer part of x , that is, the largest integer $\leq x$. Show that the two sets $A = \{\lfloor n\alpha \rfloor, n = 1, 2, \dots\}$ and $B = \{\lfloor n\beta \rfloor, n = 1, 2, \dots\}$ are disjoint and cover all the positive integers if and only if $1/\alpha + 1/\beta = 1$.
11. (*) Consider mn real numbers arranged in a $m \times n$ matrix. Sort the numbers in every one of m rows separately, so that every row contains nondecreasing numbers from left to right. Now sort the numbers in every column separately, so that every column contains nondecreasing numbers from top to bottom. The second phase may of course change things in many rows, but the point is that each row still remains sorted in nondecreasing order! Prove this. (When the numbers are interpreted as people's heights, this is known as the *marching band problem*.)
12. (*) Assume $k \geq 2$ is an integer. Prove that there exists a prime p and a strictly increasing sequence of integers a_1, a_2, \dots so that $p + ka_1, p + ka_2, \dots$ are all primes.
13. (*) A number of arcs of great circles are drawn on a sphere. (A great circle is intersection of the sphere with a plane through its center. The shortest path, or *geodesic*, between two given points on a sphere is the arc of a great circle.) The sum of lengths of these arcs is strictly less than π . Prove that there exists a plane through the center of the sphere that intersects none of these arcs.
14. Towns T_0, \dots, T_{n-1} are situated counterclockwise on a circular route of length 1. Assume that the town T_i can supply you with the amount of fuel that suffices to drive your car for distance p_i , and that $\sum_{i=0}^{n-1} p_i = 1$. That is, the total amount of fuel is exactly enough to take you once around the circle. Prove that you can start a car with an empty tank in one of the towns, drive counterclockwise around the entire circuit (picking up the fuel at each town on the route) and finish at the same spot (with an empty tank, of course).