21. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.

22. What is an indefinite integral? How do you evaluate one? What general formulas do you know for finding indefinite integrals?

23. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?

24. What is an initial value problem? How do you solve one? Give an example.

25. If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.

CHAPTER 4 Practice Exercises

Finding Extreme Values

In Exercises 1–16, find the extreme values (absolute and local) of the function over its natural domain, and where they occur.

1. $y = 2x^2 - 8x + 9$

2. $y = x^3 - 2x + 4$

3. $y = x^3 + x^2 - 8x + 5$

4. $y = x^3(x - 5)^2$

5. $y = \sqrt{x^2 - 1}$

6. $y = x - 4\sqrt{x}$

7. $y = \dfrac{1}{\sqrt[3]{1 - x^2}}$

8. $y = \sqrt{3 + 2x - x^2}$

9. $y = \dfrac{x}{x^2 + 1}$

10. $y = \dfrac{x + 1}{x^2 + 2x + 2}$

11. $y = e^x + e^{-x}$

12. $y = e^x - e^{-x}$

13. $y = x \ln x$

14. $y = x^2 \ln x$

15. $y = \cos^{-1}(x^2)$

16. $y = \sin^{-1}(e^x)$

Extreme Values

17. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.

18. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.

19. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

20. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

21. Does $g(x) = e^x - x$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of g.

22. Does $f(x) = 2e^x/(1 + x^2)$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

In Exercises 23 and 24, find the absolute maximum and absolute minimum values of f over the interval.

23. $f(x) = x - 2 \ln x, \quad 1 \le x \le 3$

24. $f(x) = (4/x) + \ln x^2, \quad 1 \le x \le 4$

25. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.

26. a. Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.

b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.

27. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?

28. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \le x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

T **29.** A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.

a. Graph f over the interval $-2.5 \le x \le 2.5$. Where does the graph appear to have local extreme values or points of inflection?

b. Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.

c. Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(*Source*: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under a grant from the National Science Foundation, USE-8950044.)

T **30.** (*Continuation of Exercise 29.*)

a. Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \le x \le 2$. Where does the graph appear to have local extreme values or points of inflection?

b. Show that f has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.

c. Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

31. a. Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

32. a. Show that $y = \tan\theta$ increases on every open interval in its domain.

b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan \pi = 0$ is less than $\tan(\pi/4) = 1$?

33. a. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.

T **b.** Find the solution to as many decimal places as you can.

34. a. Show that $f(x) = x/(x + 1)$ increases on every open interval in its domain.

b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

35. Water in a reservoir As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

36. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

37. Show that

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{d}{dx}\left(-\frac{1}{x+1}\right)$$

even though

$$\frac{x}{x+1} \neq -\frac{1}{x+1}.$$

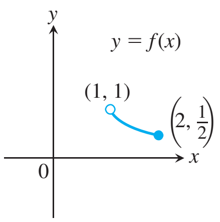
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

38. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

Analyzing Graphs

In Exercises 39 and 40, use the graph to answer the questions.

39. Identify any global extreme values of f and the values of x at which they occur.

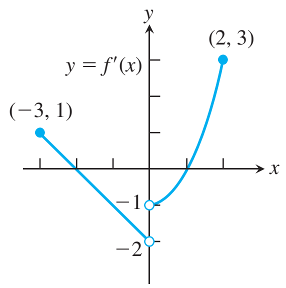


40. Estimate the open intervals on which the function $y = f(x)$ is

a. increasing.

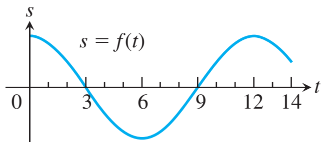
b. decreasing.

c. Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.

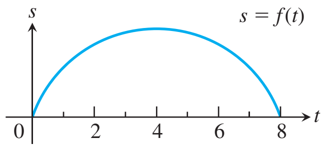


Each of the graphs in Exercises 41 and 42 is the graph of the position function $s = f(t)$ of an object moving on a coordinate line (t represents time). At approximately what times (if any) is each object's **(a)** velocity equal to zero? **(b)** Acceleration equal to zero? During approximately what time intervals does the object move **(c)** forward? **(d)** Backward?

41.



42.



Graphs and Graphing

Graph the curves in Exercises 43–58.

43. $y = x^2 - (x^3/6)$

44. $y = x^3 - 3x^2 + 3$

45. $y = -x^3 + 6x^2 - 9x + 3$

46. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

47. $y = x^3(8 - x)$

48. $y = x^2(2x^2 - 9)$

49. $y = x - 3x^{2/3}$

50. $y = x^{1/3}(x - 4)$

51. $y = x\sqrt{3 - x}$

52. $y = x\sqrt{4 - x^2}$

53. $y = (x - 3)^2 e^x$

54. $y = xe^{-x^2}$

55. $y = \ln(x^2 - 4x + 3)$

56. $y = \ln(\sin x)$

57. $y = \sin^{-1}\left(\frac{1}{x}\right)$

58. $y = \tan^{-1}\left(\frac{1}{x}\right)$

Each of Exercises 59–64 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

59. $y' = 16 - x^2$

60. $y' = x^2 - x - 6$

61. $y' = 6x(x + 1)(x - 2)$

62. $y' = x^2(6 - 4x)$

63. $y' = x^4 - 2x^2$

64. $y' = 4x^2 - x^4$

In Exercises 65–68, graph each function. Then use the function's first derivative to explain what you see.

65. $y = x^{2/3} + (x - 1)^{1/3}$

66. $y = x^{2/3} + (x - 1)^{2/3}$

67. $y = x^{1/3} + (x - 1)^{1/3}$

68. $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the rational functions in Exercises 69–76.

69. $y = \frac{x + 1}{x - 3}$

70. $y = \frac{2x}{x + 5}$

71. $y = \frac{x^2 + 1}{x}$

72. $y = \frac{x^2 - x + 1}{x}$

73. $y = \frac{x^3 + 2}{2x}$

74. $y = \frac{x^4 - 1}{x^2}$

75. $y = \frac{x^2 - 4}{x^2 - 3}$

76. $y = \frac{x^2}{x^2 - 4}$

Using L'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 77–88.

77. $\lim_{x\to 1} \frac{x^2 + 3x - 4}{x - 1}$

78. $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$

79. $\lim_{x\to \pi} \frac{\tan x}{x}$

80. $\lim_{x\to 0} \frac{\tan x}{x + \sin x}$

81. $\lim_{x\to 0} \frac{\sin^2 x}{\tan(x^2)}$

82. $\lim_{x\to 0} \frac{\sin mx}{\sin nx}$

83. $\lim_{x\to \pi/2^-} \sec 7x \cos 3x$

84. $\lim_{x\to 0^+} \sqrt{x} \sec x$

85. $\lim_{x\to 0} (\csc x - \cot x)$

86. $\lim_{x\to 0} \left(\frac{1}{x^4} - \frac{1}{x^2}\right)$

87. $\lim_{x\to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

88. $\lim_{x\to \infty} \left(\frac{x^3}{x^2 - 1} - \frac{x^3}{x^2 + 1}\right)$

Find the limits in Exercises 89–102.

89. $\lim_{x\to 0} \frac{10^x - 1}{x}$

90. $\lim_{\theta\to 0} \frac{3^\theta - 1}{\theta}$

91. $\lim_{x\to 0} \frac{2^{\sin x} - 1}{e^x - 1}$

92. $\lim_{x\to 0} \frac{2^{-\sin x} - 1}{e^x - 1}$

93. $\lim_{x\to 0} \frac{5 - 5\cos x}{e^x - x - 1}$

94. $\lim_{x\to 0} \frac{4 - 4e^x}{xe^x}$

95. $\lim_{t\to 0^+} \frac{t - \ln(1 + 2t)}{t^2}$

96. $\lim_{x\to 4} \frac{\sin^2(\pi x)}{e^{x-4} + 3 - x}$

97. $\lim_{t\to 0^+} \left(\frac{e^t}{t} - \frac{1}{t}\right)$

98. $\lim_{y\to 0^+} e^{-1/y} \ln y$

99. $\lim_{x\to \infty} \left(1 + \frac{b}{x}\right)^{kx}$

100. $\lim_{x\to \infty} \left(1 + \frac{2}{x} + \frac{7}{x^2}\right)$

101. $\lim_{x\to 0} \frac{\cos 2x - 1 - \sqrt{1 - \cos x}}{\sin^2 x}$

102. $\lim_{x\to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$

Optimization

103. The sum of two nonnegative numbers is 36. Find the numbers if

a. the difference of their square roots is to be as large as possible.

b. the sum of their square roots is to be as large as possible.

104. The sum of two nonnegative numbers is 20. Find the numbers

a. if the product of one number and the square root of the other is to be as large as possible.

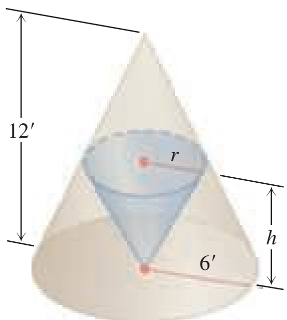
b. if one number plus the square root of the other is to be as large as possible.

105. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

106. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

107. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

108. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



109. Manufacturing tires Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

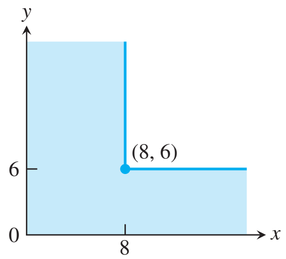
110. Particle motion The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

a. What is the farthest apart the particles ever get?

b. When do the particles collide?

T **111. Open-top box** An open-top rectangular box is constructed from a 10-in.-by-16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

112. The ladder problem What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



Newton's Method

113. Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval $[2, 3]$ and use Newton's method to find it.

114. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval $[3, 4]$ and use Newton's method to find it.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 115–138. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

115. $\int (x^3 + 5x - 7)\, dx$

116. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

117. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

118. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

119. $\int \frac{dr}{(r + 5)^2}$

120. $\int \frac{6\, dr}{(r - \sqrt{2})^3}$

121. $\int 3\theta\sqrt{\theta^2 + 1}\, d\theta$

122. $\int \frac{\theta}{\sqrt{7 + \theta^2}} d\theta$

123. $\int x^3(1 + x^4)^{-1/4}\, dx$

124. $\int (2 - x)^{3/5}\, dx$

125. $\int \sec^2 \frac{s}{10} ds$

126. $\int \csc^2 \pi s\, ds$

127. $\int \csc\sqrt{2}\theta \cot\sqrt{2}\theta\, d\theta$

128. $\int \sec\frac{\theta}{3} \tan\frac{\theta}{3}\, d\theta$

129. $\int \sin^2 \frac{x}{4}\, dx$ $\left(\textit{Hint:}\ \sin^2\theta = \frac{1 - \cos 2\theta}{2}\right)$

130. $\int \cos^2 \frac{x}{2}\, dx$

131. $\int \left(\frac{3}{x} - x\right) dx$

132. $\int \left(\frac{5}{x^2} + \frac{2}{x^2 + 1}\right) dx$

133. $\int \left(\frac{1}{2}e^t - e^{-t}\right) dt$

134. $\int (5^s + s^5)\, ds$

135. $\int \theta^{1-\pi}\, d\theta$

136. $\int 2^{\pi + r}\, dr$

137. $\int \frac{3}{2x\sqrt{x^2 - 1}} dx$

138. $\int \frac{d\theta}{\sqrt{16 - \theta^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 139–142.

139. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

140. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

141. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

142. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

Applications and Examples

143. Can the integrations in (a) and (b) both be correct? Explain.

a. $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$

b. $\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

144. Can the integrations in (a) and (b) both be correct? Explain.

a. $\int \frac{dx}{\sqrt{1 - x^2}} = -\int -\frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x + C$

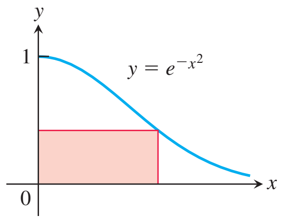
b. $\int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{-du}{\sqrt{1 - (-u)^2}}$ $\quad x = -u,\ dx = -du$

$= \int \frac{-du}{\sqrt{1 - u^2}}$

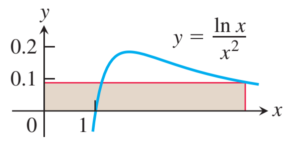
$= \cos^{-1} u + C$

$= \cos^{-1}(-x) + C$ $\quad u = -x$

145. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = e^{-x^2}$. What dimensions give the rectangle its largest area, and what is that area?



146. The rectangle shown here has one side on the positive y-axis, one side on the positive x-axis, and its upper right-hand vertex on the curve $y = (\ln x)/x^2$. What dimensions give the rectangle its largest area, and what is that area?



In Exercises 147 and 148, find the absolute maximum and minimum values of each function on the given interval.

147. $y = x\ln 2x - x, \quad \left[\frac{1}{2e}, \frac{e}{2}\right]$

148. $y = 10x(2 - \ln x), \quad (0, e^2]$

In Exercises 149 and 150, find the absolute maxima and minima of the functions and give the x-coordinates where they occur.

149. $f(x) = e^{x/\sqrt{x^4+1}}$

150. $g(x) = e^{\sqrt{3-2x-x^2}}$

T **151.** Graph the following functions and use what you see to locate and estimate the extreme values, identify the coordinates of the inflection points, and identify the intervals on which the graphs are concave up and concave down. Then confirm your estimates by working with the functions' derivatives.

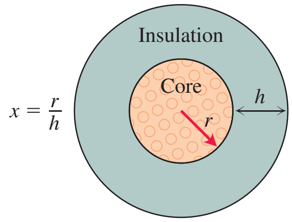
a. $y = (\ln x)/\sqrt{x}$ **b.** $y = e^{-x^2}$

c. $y = (1 + x)e^{-x}$

T **152.** Graph $f(x) = x\ln x$. Does the function appear to have an absolute minimum value? Confirm your answer with calculus.

T **153.** Graph $f(x) = (\sin x)^{\sin x}$ over $[0, 3\pi]$. Explain what you see.

154. A round underwater transmission cable consists of a core of copper wires surrounded by nonconducting insulation. If x denotes the ratio of the radius of the core to the thickness of the insulation, it is known that the speed of the transmission signal is given by the equation $v = x^2\ln(1/x)$. If the radius of the core is 1 cm, what insulation thickness h will allow the greatest transmission speed?



CHAPTER 4 Additional and Advanced Exercises

Functions and Derivatives

1. What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.

2. Is it true that a discontinuous function cannot have both an absolute maximum and an absolute minimum value on a closed interval? Give reasons for your answer.

3. Can you conclude anything about the extreme values of a continuous function on an open interval? On a half-open interval? Give reasons for your answer.

4. Local extrema Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x-1)(x-2)^2(x-3)^3(x-4)^4$$

to identify the points where f has local maximum and minimum values.

5. Local extrema

a. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6(x+1)(x-2)^2.$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

b. Suppose that the first derivative of $y = f(x)$ is

$$y' = 6x(x+1)(x-2).$$

At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

6. If $f'(x) \le 2$ for all x, what is the most the values of f can increase on $[0, 6]$? Give reasons for your answer.

7. Bounding a function Suppose that f is continuous on $[a, b]$ and that c is an interior point of the interval. Show that if $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. An inequality

a. Show that $-1/2 \le x/(1 + x^2) \le 1/2$ for every value of x.

b. Suppose that f is a function whose derivative is $f'(x) = x/(1 + x^2)$. Use the result in part (a) to show that

$$|f(b) - f(a)| \le \frac{1}{2}|b - a|$$

for any a and b.

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. Extrema and inflection points Let $h = fg$ be the product of two differentiable functions of x.

a. If f and g are positive, with local maxima at $x = a$, and if f' and g' change sign at a, does h have a local maximum at a?

b. If the graphs of f and g have inflection points at $x = a$, does the graph of h have an inflection point at a?

In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

11. Finding a function Use the following information to find the values of a, b, and c in the formula $f(x) = (x + a)/(bx^2 + cx + 2)$.

a. The values of a, b, and c are either 0 or 1.

b. The graph of f passes through the point $(-1, 0)$.

c. The line $y = 1$ is an asymptote of the graph of f.

12. Horizontal tangent For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

Optimization

13. Largest inscribed triangle Points A and B lie at the ends of a diameter of a unit circle and point C lies on the circumference. Is it true that the area of triangle ABC is largest when the triangle is isosceles? How do you know?

14. Proving the second derivative test The Second Derivative Test for Local Maxima and Minima (Section 4.4) says: