Math 21A–C, Fall 2005.
Dec. 12, 2005.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Problem 1 is worth 20 points, and problems 2 through 7 are each worth 30 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 10 pages (including this one) with 7 problems. Note that problems 4 and 7 are each given on two pages. Read through the entire exam before beginning to work.

1	
2	
3	
4	
5	
6	
7	
TOTAL	

3. Consider the function

$$f(x) = \begin{cases} -3x^2, & x < -1, \\ ax + b, & -1 \le x \le 1, \\ x^2, & x > 1. \end{cases} \qquad 2x - 1$$

(a) Determine the numbers a and b so that $y = f(x)$ is continuous for all x.

Continuity at 1: $a + b = 1$

Continuity at -1: $-a + b = -3$

$2b = -2$, $\underline{b = -1}$, $\underline{a = 1 - b = 2}$

(b) Assume a and b are determined as in (a). Is $y = f(x)$ differentiable at $x = 1$?

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^+} \frac{\cancel{1} + 2h + h^2 - \cancel{1}}{h}$$

$$= \lim_{h \to 0^+} \frac{\cancel{h}(2+h)}{\cancel{h}} = 2$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2(\cancel{1}+h) \cancel{-1} - \cancel{1}}{h} = 2$$

The two limits are equal: $\boxed{\text{YES}}$.

1. Compute the following limits, *in any correct way* you can. Give each answer as a finite number, $+\infty$ or $-\infty$.

(a) $\displaystyle\lim_{x\to 2^+} \frac{x^2-9x+3}{\ln(x-1)} = \underline{\underline{-\infty}}$

$\ln(x-1)$ is small and >0

x^2-9x+3 is close to -11

(b) $\displaystyle\lim_{x\to 0} \frac{e^{2x}-2x-1}{x^2} = \lim_{x\to 0} \frac{2e^{2x}-2}{2x} = \lim_{x\to 0} \frac{4e^{2x}}{2} = \underline{\underline{2}}$

$\left(\frac{0}{0}\right)$, L'Hopital $\qquad$ $\left(\frac{0}{0}\right)$, L'Hopital

(c) (Here $x>0$.) $\displaystyle\lim_{h\to 0} \frac{(x+h)^{3/4}-x^{3/4}}{h} = f'(x) = \underline{\underline{\frac{3}{4}x^{-1/4}}}$

$f(x) = x^{3/4}$

(d) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right) = \lim_{x\to\infty} \frac{\cancel{x^2}+2x-(\cancel{x^2}-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$

$\displaystyle = \lim_{x\to\infty} \frac{4x \;/x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}\;/x} = \lim_{x\to\infty} \frac{4}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}}$

$= \underline{\underline{2}}$

2.
(a) $f(x) = x^4 \cdot \sqrt{1-x^3}$ Compute $f'(x)$. *Do not simplify.*

$$f'(x) = 4x^3\sqrt{1-x^3} + x^4 \cdot \tfrac{1}{2}(1-x^3)^{-1/2} \cdot (-3x^2)$$

(b) $f(x) = x^2 \tan(3x)$. Compute $f'(x)$. *Do not simplify.*

$$f'(x) = 2x\tan(3x) + x^2 \cdot \frac{1}{\cos^2 3x} \cdot 3$$

(c) Assume that $f(x)$ satisfies the equation $(x + f(x))^4 + xf(x) - f(x) = 0$ and that $f(0) = 1$. Compute $f'(0)$.

$$4(x+f(x))^3(1+f'(x)) + f(x) + xf'(x) - f'(x) = 0$$

$x = 0$, $f(0) = 1$:

$$4(1+f'(0)) + 1 - f'(0) = 0$$

$$5 + 3f'(0) = 0$$

$$\underline{\underline{f'(0) = -\frac{5}{3}}}$$

4. Throughout this problem, $f(x) = \dfrac{9(x^2-3)}{x^3} = 9(x^{-1} - 3x^{-3})$.

Some help with computations below: use $\sqrt{3} \approx 1.7$, $\sqrt{18} = 3\sqrt{2} \approx 4$, $f(3\sqrt{2}) \approx 1.8$.

(a) Is this function odd or even?

<u>Odd</u>: $f(-x) = \dfrac{9(x^2-3)}{-x^3} = -f(x)$

(b) Determine the domain of $y = f(x)$, its intercepts, and horizontal and vertical asymptotes.

Domain: $x \neq 0$

Intercepts: $(\sqrt{3}, 0), (-\sqrt{3}, 0)$

$\lim_{x\to\infty} f(x) = 0$ $\quad y=0$ horizontal asymptote

$\lim_{x\to 0+} f(x) = -\infty$, $\lim_{x\to 0-} f(x) = +\infty$ $\quad x=0$ vertical asymptote

(c) Determine the intervals on which $y = f(x)$ is increasing and the intervals on which it is decreasing.

$f'(x) = 9(-x^{-2} + 9x^{-4}) = -9x^{-4}(x^2-9)$

Critical pts.: $x = 0, 3, -3$

	$(-\infty,-3)$	$(-3,0)$	$(0,3)$	$(3,\infty)$
sign of f'	$-$	$+$	$+$	$-$
	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$

$(-3,-2)$ local min. $\qquad (3,2)$ local max.

(d) Determine the intervals on which $y = f(x)$ is concave up and the intervals on which it is concave down.

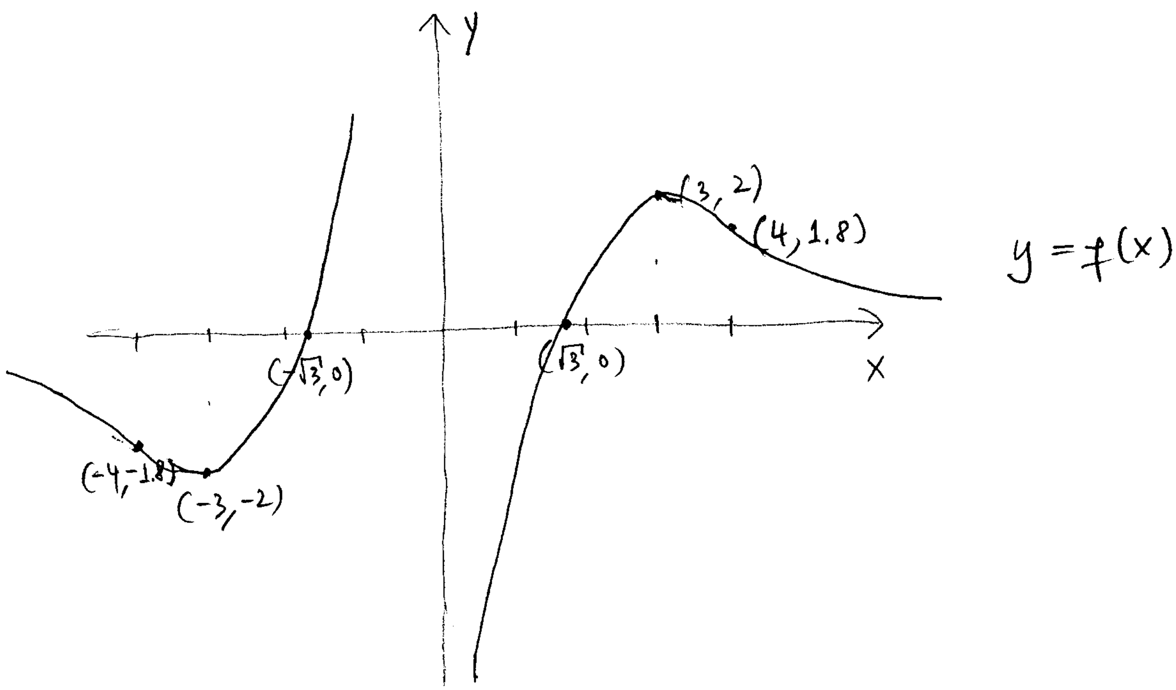
$f''(x) = 9(+2x^{-3} - 36x^{-5}) = 18x^{-5}(x^2-18)$

$f''(0) = 0$ at $x = \pm\sqrt{18}$, undefined at $x=0$.

	$(\infty, -\sqrt{18})$	$(-\sqrt{18}, 0)$	$(0, \sqrt{18})$	$(\sqrt{18}, 0)$
sign of f''	$-$	$+$	$-$	$+$
	$\frown$	$\smile$	$\frown$	$\smile$

pts. of inflection at $x = \sqrt{18}, -\sqrt{18}$.

Problem 4, continued.
(e) Sketch the graph of $y = f(x)$. Label clearly all local maxima and minima, and inflection points.



(f) Let $g(x) = \sqrt{x}$. Determine the domain and range of the composite function $f(g(x))$. (No further computations are necessary for this!)

Domain: $x > 0$, i.e., $(0, \infty)$

Range: $(-\infty, 2]$

5. Provide straightforward, and fully justified, answers to the following questions. In each of them, assume that $f(x)$ is a continuous function defined for all x, and $f'(x)$ and $f''(x)$ exist and are continuous for all x.

(a) $f''(x) = \dfrac{1+3x^2+5x^4}{1+7(\sin x)^2}$. Can f have a local maximum?

$\boxed{\text{No.}}$ $f''(x) > 0$ for all x, so any critical point is a local minimum.

(b) Let $f(x) = 5x - 4\cos x$. How many x–intercepts does f have? $\boxed{1}$

$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$, and f is continuous, so there is at least one x-intercept by IVT.

$f'(x) = 5 + 4\sin x \geq 1 > 0$. f is increasing so that there is at most one x-intercept.

(c) $f(0) = 5, f''(0) = 1$. Is it possible that $f(x) \leq 5$ for all x?

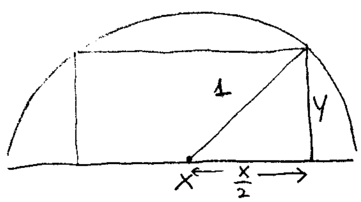
$\boxed{\text{No.}}$ If $f(x) \leq 5$ for all x, then f has a local max at $x=0$, so that $f'(0) = 0$. But ~~the~~ then there must be a local min at $x=0$ since $f''(0) > 0$. This is impossible.

(d) $f(1) = f(3) = f(5) = 0$. Must there be an x for which $f''(x) = 0$?

There must be c_1 in $(1,3)$ so that $f'(c_1) = 0$.
there must be c_2 in $(3,5)$ so that $f'(c_2) = 0$.

Then there must be a number c between c_1 and c_2 so that $f''(c) = 0$. $\boxed{\text{YES}}$

6. You have a piece of paper in the shape of semi–circle with radius 1cm. You wish to cut a rectangle from this piece of paper so that one side of the rectangle is along the diameter. Find the dimensions of the rectangle with the largest possible area.



$$A = xy$$

$$\frac{x^2}{4} + y^2 = 1$$

$$y = \sqrt{1 - \frac{x^2}{4}}$$

$$0 \leq x \leq 2$$

$$A = x\sqrt{1 - \frac{x^2}{4}}$$

$$\frac{dA}{dx} = \sqrt{1 - \frac{x^2}{4}} + x \cdot \frac{1}{\cancel{2}} \cdot \left(1 - \frac{x^2}{4}\right)^{-1/2} \cdot \frac{-\cancel{2}x}{4} = 0$$

$$1 - \frac{x^2}{4} + \frac{x^2}{4} = 0$$

$$1 - \frac{x^2}{2} = 0$$

$$\boxed{\begin{array}{l} x = \sqrt{2} \\ y = \frac{\sqrt{2}}{2} \end{array}}$$

critical pt. This must be max, as $A = 0$ at $x = 0$ and $x = 2$.

7. Let $f(x) = \frac{1}{2}x^{3/2}$ throughout this problem. Restrict x to $[0, \infty)$, the domain of this function.
(a) Is $y = f(x)$ one-to-one?

$$f'(x) = \frac{3}{4}x^{1/2} > 0 \quad \text{when} \quad x > 0.$$

So f is increasing, so one-to-one. $\boxed{\text{YES}}$

(b) Find the point on the graph which is closest to $(2,0)$. (Don't forget that it is enough to minimize the *square* of the distance between $(2,0)$ and a point (x,y) on the graph! Also note that $64 + 4 \cdot 3 \cdot 16 = 16^2$.)

$$g(x) = D^2 = (x-2)^2 + \frac{1}{4}x^3$$

$$g'(x) = 2(x-2) + \frac{3}{4}x^2 = 0$$

$$3x^2 + 8x - 16 = 0$$

$$x = \frac{-8 \pm \sqrt{64 + 4 \cdot 3 \cdot 16}}{6} = \frac{-8 \oplus 16}{6} = \frac{4}{3}$$

	$(0, \frac{4}{3})$	$(\frac{4}{3}, \infty)$
sign of g'	$-$	$+$
	$\searrow$	$\nearrow$

Global min at $x = \frac{4}{3}$

Point on the graph: $\boxed{\left(\frac{4}{3}, \frac{4}{3\sqrt{3}}\right)}$

Problem 7, continued (still with $f(x) = \frac{1}{2}x^{3/2}$).
(c) A particle is moving on the graph of $y = f(x)$. Its position is described by (x, y), where x and y are both functions of time t. At one instant, you observe that $x = 2$ and $\frac{dx}{dt} = 5$. At what rate is the distance between the particle and point $(2, 0)$ changing at this instant?

$$D^2 = (x-2)^2 + \frac{1}{4}x^3$$

When $x = 2$, $D^2 = 2$, $D = \sqrt{2}$.

$$2D \cdot \frac{dD}{dt} = 2(x-2)\frac{dx}{dt} + \frac{3}{4}x^2 \frac{dx}{dt}$$

Plug in $x=2$, $\frac{dx}{dt} = 5$, $D = \sqrt{2}$,

$$2\sqrt{2}\,\frac{dD}{dt} = 3 \cdot 5 \qquad \boxed{\frac{dD}{dt} = \frac{15}{2\sqrt{2}}}$$

(d) Find a point on the graph of $y = f(x)$ at which the tangent line goes through the point $(2, 0)$.

Point : $(a, \frac{1}{2}a^{3/2})$

Slope : $\frac{3}{4}a^{1/2}$

Line : $\frac{3}{4}a^{1/2}(x-a) = y - \frac{1}{2}a^{3/2}$

Plug in $x = 2$, $y = 0$:

$$\frac{3}{4}a^{1/2}(2-a) = -\frac{1}{2}a^{3/2} \qquad \cdot \frac{4}{a^{1/2}}$$

$$3(2-a) = -2a$$

$$6 - 3a = -2a \qquad \boxed{a = 6}$$

$$\boxed{\text{Pt. } (6, \tfrac{1}{2}6^{3/2})}$$