Math 21A–C, Fall 2005.
Oct. 26, 2005.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, first name first): KEY

NAME(sign): ______________________

ID#: ______________________

Instructions: Each of the four problems is worth 25 points. Read each question carefully and answer it in the space provided. YOU MUST SHOW ALL YOUR WORK TO RECEIVE FULL CREDIT. Clarity of your solutions may be a factor in determining credit. Calculators, books or notes are not allowed.

Make sure that you have a total of 5 pages (including this one) with 4 problems. Read through the entire exam before beginning to work.

1	
2	
3	
4	
TOTAL	

1. Below are 2 questions. Provide straightforward answers with clear explanations.
(a) Can a be chosen so that

$$f(x) = \begin{cases} \dfrac{x^4 - 2x^3}{x-2}, & \text{if } x \neq 2, \\ a, & \text{if } x = 2 \end{cases}$$

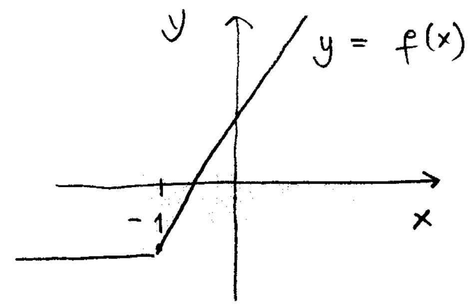
is a continuous function at every x?

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^4 - 2x^3}{x-2} = \lim_{x\to 2} \frac{x^3\cancel{(x-2)}}{\cancel{x-2}} = \underline{\underline{8}}$$

Yes, $\boxed{a = 8}$.

(b) Is the function $f(x) = x + |x+1|$ continuous at every x? Is it differentiable at every x? Sketch the graph of $y = f(x)$.

$$f(x) = \begin{cases} 2x+1, & x \geq -1 \\ -1, & x \leq -1 \end{cases}$$



$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^-} f(x) = -1$$, so f <u>is</u> continuous for every x.

$$\lim_{h\to 0^+} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0+} \frac{2(-1+h)+1-(-1)}{h} = 2$$

$$\lim_{h\to 0^-} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0^-} \frac{(-1)-(-1)}{h} = 0$$

$y = f(x)$ is <u>not</u> differentiable at $x = -1$.

2. Consider the function $f(x) = \dfrac{x^2 - x}{x^2 - 4}$. Determine the intercepts, and vertical and horizontal asymptotes. Determine also any points where the graph of $y = f(x)$ intersects its horizontal asymptote. Then sketch the graph of this function on which all obtained points and asymptotes are clearly marked.

$f(x) = \dfrac{x(x-1)}{(x-2)(x+2)}$ Intercepts: $\underline{(0,0)}$, $\underline{(1,0)}$

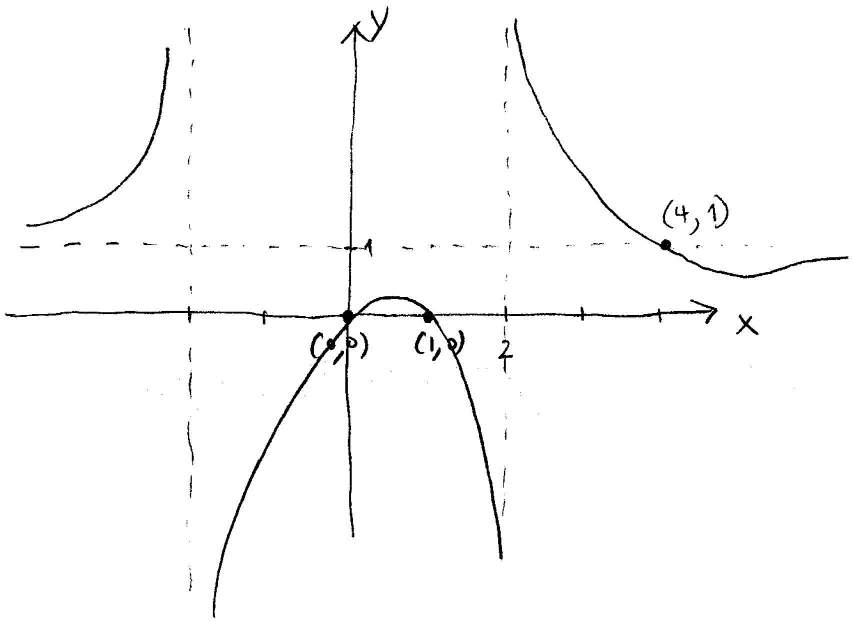
$\lim_{x \to \pm\infty} f(x) = 1$ $\underline{y=1}$ horizontal asymptote

$\underline{x=2}$, $\underline{x=-2}$ vertical asymptotes

$\lim_{x \to 2^+} f(x) = \infty$ $\lim_{x \to 2^-} f(x) = -\infty$ $\lim_{x \to -2^+} f(x) = -\infty$ $\lim_{x \to -2^-} f(x) = +\infty$

Intersection with horizontal asymptote:

$f(x) = 1$, $x^2 - x = x^2 - 4$, $x = 4$, $\underline{(4, 1)}$



3. Compute the following limits. Give each answer as a finite number, $+\infty$ or $-\infty$.

(a) $\lim_{x\to 0} \frac{\sin(6x)}{\sin(2x)\cos(4x)}$ $= \lim_{x\to 0} \underbrace{\frac{\sin 6x}{6x}}_{\to 1} \cdot \underbrace{\frac{2x}{\sin 2x}}_{\to 1} \cdot \underbrace{\frac{1}{\cos 4x}}_{\to 1} \cdot \frac{6x}{2x} = \underline{\underline{3}}$

(b) $\lim_{x\to 4} \frac{x-4}{x-3-\sqrt{x-3}}$ $= \lim_{x\to 4} \frac{x-4}{(x-3)^2-(x-3)} \cdot (x-3+\sqrt{x-3})$

$= \lim_{x\to 4} \frac{\cancel{x-4}}{\underbrace{(x-3)}_{\to 1}\cancel{(x-4)}} \cdot \underbrace{(x-3+\sqrt{x-3})}_{\to 2}$

$= \underline{\underline{2}}$

(c) $\lim_{x\to\infty} \frac{x-4}{3x+\sqrt{x}+4}$ $\begin{matrix} /x \\ /x \end{matrix}$ $= \lim_{x\to\infty} \frac{1-\frac{4}{x}}{3+\frac{1}{\sqrt{x}}+\frac{4}{x}} = \underline{\underline{\frac{1}{3}}}$

(d) $\lim_{x\to 1^+} \frac{x^2-9}{\sqrt{x+3}-2}$ $= \underline{\underline{-\infty}}$

correct!

$x^2 - 9 \to -8$

$\sqrt{x+3} - 2$ is small and > 0

4. Let $f(x) = \dfrac{2x+3}{3x+2}$.

(a) Determine $L = \lim_{x\to\infty} f(x)$. Determine how large a B you should take in order to ensure that $|f(x) - L| < 0.001$ for $x > B$.

Call $\varepsilon = 0.001$.

$$L = \lim_{x\to\infty} \frac{2 + 3/x}{3 + 2/x} = \frac{2}{3}.$$

$$\left| f(x) - \frac{2}{3} \right| < \varepsilon$$

$$\left| \frac{2x+3}{3x+2} - \frac{2}{3} \right| < \varepsilon$$

$$\left| \frac{\cancel{6x} + 9 - \cancel{6x} - 4}{3(3x+2)} \right| < \varepsilon$$

(Note. x can be assumed positive.)

$$\frac{5}{3} \cdot \frac{1}{3x+2} < \varepsilon$$

$$3x + 2 > \frac{5}{3} \cdot \frac{1}{\varepsilon} \qquad 3x > \frac{5}{3} \cdot \frac{1}{\varepsilon} - 2, \qquad x > \frac{5}{9} \cdot \frac{1}{\varepsilon} - \frac{2}{3}$$

Can take $B = \frac{5}{9} \cdot 1000 - \frac{2}{3}$, or anything larger.

(b) Let $g(x) = x^4 + 1$. Do graphs of $y = f(x)$ and $y = g(x)$ intersect for some *positive* x?

$$h(x) = \frac{2x+3}{3x+2} - (x^4 + 1), \quad \text{continuous for } x \geq 0.$$

$$h(0) = \frac{3}{2} - 1 = \frac{1}{2} > 0$$

$$h(1) = 1 - 2 = -1 < 0$$

By IVT, $h(x) = 0$ for some x between 0 and 1.

Yes.

(c) Determine the slope of the tangent to the graph of $y = f(x)$ at the point at which it intersects the x-axis.

$$f'(x) = \frac{2(3x+2) - 3(2x+3)}{(3x+2)^2} = \frac{-5}{(3x+2)^2}$$

The pt. is $(-3/2, 0)$. Slope at this pt.: $\dfrac{-5}{(-5/2)^2} = -\dfrac{4}{5}$

Line: $y - 0 = -\frac{4}{5}\left(x + \frac{3}{2}\right)$; $y = -\frac{4}{5}x - \frac{6}{5}$