Chapter 11 Questions to Guide Your Review

1. What is a parametrization of a curve in the xy-plane? Does a function $y = f(x)$ always have a parametrization? Are parametrizations of a curve unique? Give examples.
2. Give some typical parametrizations for lines, circles, parabolas, ellipses, and hyperbolas. How might the parametrized curve differ from the graph of its Cartesian equation?
3. What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids?
4. What is the formula for the slope dy/dx of a parametrized curve $x = f(t)$, $y = g(t)$? When does the formula apply? When can you expect to be able to find d^2y/dx^2 as well? Give examples.
5. How can you sometimes find the area bounded by a parametrized curve and one of the coordinate axes?
6. How do you find the length of a smooth parametrized curve $x = f(t)$, $y = g(t)$, $a \le t \le b$? What does smoothness have to do with length? What else do you need to know about the parametrization in order to find the curve's length? Give examples.
7. What is the arc length function for a smooth parametrized curve? What is its arc length differential?
8. Under what conditions can you find the area of the surface generated by revolving a curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, about the x-axis? the y-axis? Give examples.
9. What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other?
10. What consequence does the lack of uniqueness of polar coordinates have for graphing? Give an example.
11. How do you graph equations in polar coordinates? Include in your discussion symmetry, slope, behavior at the origin, and the use of Cartesian graphs. Give examples.
12. How do you find the area of a region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give examples.
13. Under what conditions can you find the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give an example of a typical calculation.
14. What is a parabola? What are the Cartesian equations for parabolas whose vertices lie at the origin and whose foci lie on the coordinate axes? How can you find the focus and directrix of such a parabola from its equation?
15. What is an ellipse? What are the Cartesian equations for ellipses centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
16. What is a hyperbola? What are the Cartesian equations for hyperbolas centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
17. What is the eccentricity of a conic section? How can you classify conic sections by eccentricity? How does eccentricity change the shape of ellipses and hyperbolas?
18. Explain the equation $PF = e \cdot PD$.
19. What are the standard equations for lines and conic sections in polar coordinates? Give examples.

Chapter 11 Practice Exercises

Identifying Parametric Equations in the Plane

Exercises 1–6 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation and indicate the direction of motion and the portion traced by the particle.

1. $x = t/2, \quad y = t + 1; \quad -\infty < t < \infty$
2. $x = \sqrt{t}, \quad y = 1 - \sqrt{t}; \quad t \ge 0$
3. $x = (1/2)\tan t, \quad y = (1/2)\sec t; \quad -\pi/2 < t < \pi/2$
4. $x = -2\cos t, \quad y = 2\sin t; \quad 0 \le t \le \pi$
5. $x = -\cos t, \quad y = \cos^2 t; \quad 0 \le t \le \pi$
6. $x = 4\cos t, \quad y = 9\sin t; \quad 0 \le t \le 2\pi$

Finding Parametric Equations and Tangent Lines

7. Find parametric equations and a parameter interval for the motion of a particle in the xy-plane that traces the ellipse $16x^2 + 9y^2 = 144$ once counterclockwise. (There are many ways to do this.)
8. Find parametric equations and a parameter interval for the motion of a particle that starts at the point $(-2, 0)$ in the xy-plane and traces the circle $x^2 + y^2 = 4$ three times clockwise. (There are many ways to do this.)

In Exercises 9 and 10, find an equation for the line in the xy-plane that is tangent to the curve at the point corresponding to the given value of t. Also, find the value of d^2y/dx^2 at this point.

9. $x = (1/2)\tan t, \quad y = (1/2)\sec t; \quad t = \pi/3$
10. $x = 1 + 1/t^2, \quad y = 1 - 3/t; \quad t = 2$
11. Eliminate the parameter to express the curve in the form $y = f(x)$.
    a. $x = 4t^2, \quad y = t^3 - 1$
    b. $x = \cos t, \quad y = \tan t$
12. Find parametric equations for the given curve.
    a. Line through $(1, -2)$ with slope 3
    b. $(x - 1)^2 + (y + 2)^2 = 9$
    c. $y = 4x^2 - x$
    d. $9x^2 + 4y^2 = 36$

Lengths of Curves

Find the lengths of the curves in Exercises 13–19.

13. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

14. $x = y^{2/3}, \quad 1 \le y \le 8$

15. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

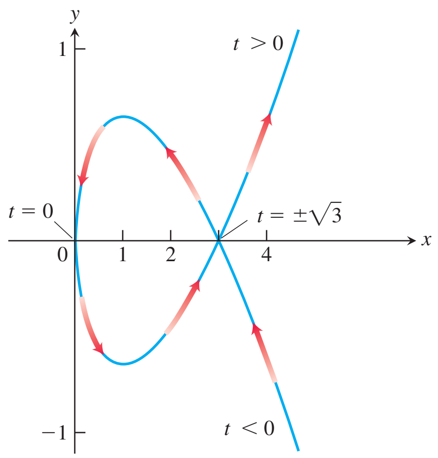
16. $x = \left(y^3/12\right) + (1/y), \quad 1 \le y \le 2$

17. $x = 5\cos t - \cos 5t, \quad y = 5\sin t - \sin 5t, \quad 0 \le t \le \pi/2$

18. $x = t^3 - 6t^2, \quad y = t^3 + 6t^2, \quad 0 \le t \le 1$

19. $x = 3\cos\theta, \quad y = 3\sin\theta, \quad 0 \le \theta \le \dfrac{3\pi}{2}$

20. Find the length of the enclosed loop $x = t^2, y = \left(t^3/3\right) - t$ shown here. The loop starts at $t = -\sqrt{3}$ and ends at $t = \sqrt{3}$.



Surface Areas

Find the areas of the surfaces generated by revolving the curves in Exercises 21 and 22 about the indicated axes.

21. $x = t^2/2, \quad y = 2t, \quad 0 \le t \le \sqrt{5}; \quad x\text{-axis}$

22. $x = t^2 + 1/(2t), \quad y = 4\sqrt{t}, \quad 1/\sqrt{2} \le t \le 1; \quad y\text{-axis}$

Polar to Cartesian Equations

Sketch the lines in Exercises 23–28. Also, find a Cartesian equation for each line.

23. $r\cos\left(\theta + \dfrac{\pi}{3}\right) = 2\sqrt{3}$ **24.** $r\cos\left(\theta - \dfrac{3\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$

25. $r = 2\sec\theta$ **26.** $r = -\sqrt{2}\sec\theta$

27. $r = -(3/2)\csc\theta$ **28.** $r = \left(3\sqrt{3}\right)\csc\theta$

Find Cartesian equations for the circles in Exercises 29–32. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

29. $r = -4\sin\theta$ **30.** $r = 3\sqrt{3}\sin\theta$

31. $r = 2\sqrt{2}\cos\theta$ **32.** $r = -6\cos\theta$

Cartesian to Polar Equations

Find polar equations for the circles in Exercises 33–36. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

33. $x^2 + y^2 + 5y = 0$ **34.** $x^2 + y^2 - 2y = 0$

35. $x^2 + y^2 - 3x = 0$ **36.** $x^2 + y^2 + 4x = 0$

Graphs in Polar Coordinates

Sketch the regions defined by the polar coordinate inequalities in Exercises 37 and 38.

37. $0 \le r \le 6\cos\theta$ **38.** $-4\sin\theta \le r \le 0$

Match each graph in Exercises 39–46 with the appropriate equation (a)–(l). There are more equations than graphs, so some equations will not be matched.

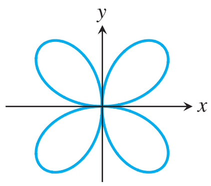
a. $r = \cos 2\theta$ **b.** $r\cos\theta = 1$ **c.** $r = \dfrac{6}{1 - 2\cos\theta}$

d. $r = \sin 2\theta$ **e.** $r = \theta$ **f.** $r^2 = \cos 2\theta$

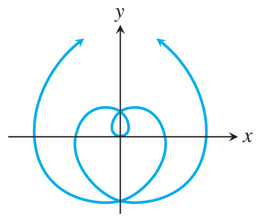
g. $r = 1 + \cos\theta$ **h.** $r = 1 - \sin\theta$ **i.** $r = \dfrac{2}{1 - \cos\theta}$

j. $r^2 = \sin 2\theta$ **k.** $r = -\sin\theta$ **l.** $r = 2\cos\theta + 1$

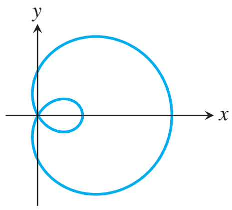
39. Four-leaved rose



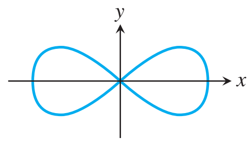
40. Spiral



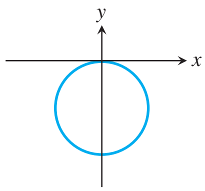
41. Limaçon



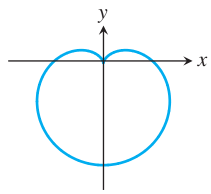
42. Lemniscate



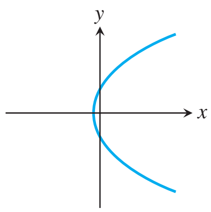
43. Circle



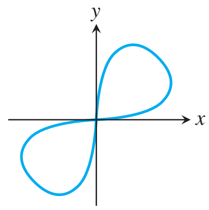
44. Cardioid



45. Parabola



46. Lemniscate



Area in Polar Coordinates

Find the areas of the regions in the polar coordinate plane described in Exercises 47–50.

47. Enclosed by the limaçon $r = 2 - \cos\theta$

48. Enclosed by one leaf of the three-leaved rose $r = \sin 3\theta$

49. Inside the "figure eight" $r = 1 + \cos 2\theta$ and outside the circle $r = 1$

50. Inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2\sin\theta$

Length in Polar Coordinates

Find the lengths of the curves given by the polar coordinate equations in Exercises 51–54.

51. $r = -1 + \cos\theta$

52. $r = 2\sin\theta + 2\cos\theta, \quad 0 \le \theta \le \pi/2$

53. $r = 8\sin^3(\theta/3), \quad 0 \le \theta \le \pi/4$

54. $r = \sqrt{1 + \cos 2\theta}, \quad -\pi/2 \le \theta \le \pi/2$

Graphing Conic Sections

Sketch the parabolas in Exercises 55–58. Include the focus and directrix in each sketch.

55. $x^2 = -4y$ **56.** $x^2 = 2y$

57. $y^2 = 3x$ **58.** $y^2 = -(8/3)x$

Find the eccentricities of the ellipses and hyperbolas in Exercises 59–62. Sketch each conic section. Include the foci, vertices, and asymptotes (as appropriate) in your sketch.

59. $16x^2 + 7y^2 = 112$ **60.** $x^2 + 2y^2 = 4$

61. $3x^2 - y^2 = 3$ **62.** $5y^2 - 4x^2 = 20$

Exercises 63–68 give equations for conic sections and tell how many units up or down and to the right or left each curve is to be shifted. Find an equation for the new conic section, and find the new foci, vertices, centers, and asymptotes, as appropriate. If the curve is a parabola, find the new directrix as well.

63. $x^2 = -12y$, right 2, up 3 **64.** $y^2 = 10x$, left 1/2, down 1

65. $\frac{x^2}{9} + \frac{y^2}{25} = 1$, left 3, down 5

66. $\frac{x^2}{169} + \frac{y^2}{144} = 1$, right 5, up 12

67. $\frac{y^2}{8} - \frac{x^2}{2} = 1$, right 2, up $2\sqrt{2}$

68. $\frac{x^2}{36} - \frac{y^2}{64} = 1$, left 10, down 3

Identifying Conic Sections

Complete the squares to identify the conic sections in Exercises 69–76. Find their foci, vertices, centers, and asymptotes (as appropriate). If the curve is a parabola, find its directrix as well.

69. $x^2 - 4x - 4y^2 = 0$ **70.** $4x^2 - y^2 + 4y = 8$

71. $y^2 - 2y + 16x = -49$ **72.** $x^2 - 2x + 8y = -17$

73. $9x^2 + 16y^2 + 54x - 64y = -1$

74. $25x^2 + 9y^2 - 100x + 54y = 44$

75. $x^2 + y^2 - 2x - 2y = 0$ **76.** $x^2 + y^2 + 4x + 2y = 1$

Conics in Polar Coordinates

Sketch the conic sections whose polar coordinate equations are given in Exercises 77–80. Give polar coordinates for the vertices and, in the case of ellipses, for the centers as well.

77. $r = \frac{2}{1 + \cos\theta}$ **78.** $r = \frac{8}{2 + \cos\theta}$

79. $r = \frac{6}{1 - 2\cos\theta}$ **80.** $r = \frac{12}{3 + \sin\theta}$

Exercises 81–84 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

81. $e = 2, \quad r\cos\theta = 2$

82. $e = 1, \quad r\cos\theta = -4$

83. $e = 1/2, \quad r\sin\theta = 2$

84. $e = 1/3, \quad r\sin\theta = -6$

Theory and Examples

85. Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about **(a)** the x-axis, **(b)** the y-axis.

86. The "triangular" region in the first quadrant bounded by the x-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

87. Show that the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the polar equation

$$r = \frac{k}{1 + e\cos\theta}$$

into the Cartesian equation

$$(1 - e^2)x^2 + y^2 + 2kex - k^2 = 0.$$

88. Archimedes spirals The graph of an equation of the form $r = a\theta$, where a is a nonzero constant, is called an *Archimedes spiral*. Is there anything special about the widths between the successive turns of such a spiral?

Chapter 11 Additional and Advanced Exercises

Finding Conic Sections

1. Find an equation for the parabola with focus (4, 0) and directrix $x = 3$. Sketch the parabola together with its vertex, focus, and directrix.

2. Find the vertex, focus, and directrix of the parabola

$$x^2 - 6x - 12y + 9 = 0.$$

3. Find an equation for the curve traced by the point $P(x, y)$ if the distance from P to the vertex of the parabola $x^2 = 4y$ is twice the distance from P to the focus. Identify the curve.

4. A line segment of length $a + b$ runs from the x-axis to the y-axis. The point P on the segment lies a units from one end and b units from the other end. Show that P traces an ellipse as the ends of the segment slide along the axes.

5. The vertices of an ellipse of eccentricity 0.5 lie at the points $(0, \pm 2)$. Where do the foci lie?

6. Find an equation for the ellipse of eccentricity $2/3$ that has the line $x = 2$ as a directrix and the point (4, 0) as the corresponding focus.