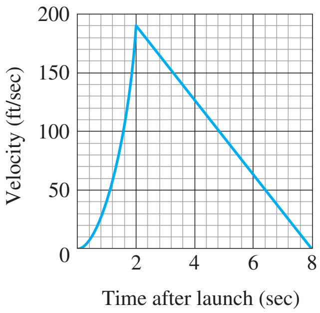
Chapter 5 Questions to Guide Your Review

1. How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
2. What is sigma notation? What advantage does it offer? Give examples.
3. What is a Riemann sum? Why might you want to consider such a sum?
4. What is the norm of a partition of a closed interval?
5. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
6. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
7. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
8. Describe the rules for working with definite integrals (Table 5.6). Give examples.
9. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
10. What is the Net Change Theorem? What does it say about the integral of velocity? The integral of marginal cost?
11. Discuss how the processes of integration and differentiation can be considered as "inverses" of each other.
12. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x)$, $y(x_0) = y_0$, when f is continuous?
13. How is integration by substitution related to the Chain Rule?
14. How can you sometimes evaluate indefinite integrals by substitution? Give examples.
15. How does the method of substitution work for definite integrals? Give examples.
16. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

Chapter 5 Practice Exercises

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

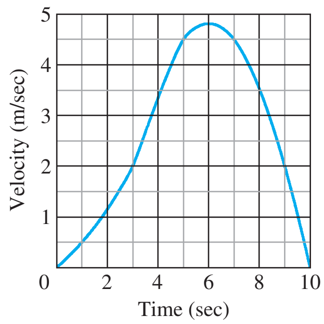


   a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.3, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)

   b. Sketch a graph of the rocket's height above ground as a function of time for $0 \le t \le 8$.

2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

   b. Sketch a graph of s as a function of t for $0 \le t \le 10$, assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

   a. $\sum_{k=1}^{10} \frac{a_k}{4}$ b. $\sum_{k=1}^{10} (b_k - 3a_k)$

   c. $\sum_{k=1}^{10} (a_k + b_k - 1)$ d. $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

   a. $\sum_{k=1}^{20} 3a_k$ b. $\sum_{k=1}^{20} (a_k + b_k)$

   c. $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$ d. $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P.

5. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k - 1)^{-1/2}\, \Delta x_k$, where P is a partition of $[1, 5]$

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k(c_k{}^2 - 1)^{1/3}\, \Delta x_k$, where P is a partition of $[1, 3]$

7. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\cos\left(\frac{c_k}{2}\right)\right) \Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin c_k)(\cos c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^{2} 3f(x)\,dx = 12$, $\int_{-2}^{5} f(x)\,dx = 6$, and $\int_{-2}^{5} g(x)\,dx = 2$, find the values of the following.

a. $\int_{-2}^{2} f(x)\,dx$ **b.** $\int_{2}^{5} f(x)\,dx$

c. $\int_{5}^{-2} g(x)\,dx$ **d.** $\int_{-2}^{5} (-\pi g(x))\,dx$

e. $\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right) dx$

10. If $\int_{0}^{2} f(x)\,dx = \pi$, $\int_{0}^{2} 7g(x)\,dx = 7$, and $\int_{0}^{1} g(x)\,dx = 2$, find the values of the following.

a. $\int_{0}^{2} g(x)\,dx$ **b.** $\int_{1}^{2} g(x)\,dx$

c. $\int_{2}^{0} f(x)\,dx$ **d.** $\int_{0}^{2} \sqrt{2}\, f(x)\,dx$

e. $\int_{0}^{2} (g(x) - 3f(x))\,dx$

Area

In Exercises 11–14, find the total area of the region between the graph of f and the x-axis.

11. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$

12. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$

13. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

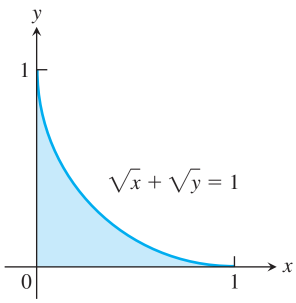
14. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

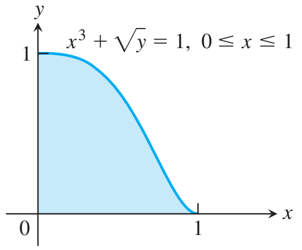
15. $y = x, \quad y = 1/x^2, \quad x = 2$

16. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$

17. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



18. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



19. $x = 2y^2, \quad x = 0, \quad y = 3$ **20.** $x = 4 - y^2, \quad x = 0$

21. $y^2 = 4x, \quad y = 4x - 2$

22. $y^2 = 4x + 4, \quad y = 4x - 16$

23. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$

24. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \le x \le \pi/2$

25. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

26. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$

27. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

28. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

29. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 3\pi/2$.

33. **Area** Find the area between the curve $y = 2(\ln x)/x$ and the x-axis from $x = 1$ to $x = e$.

34. **a.** Show that the area between the curve $y = 1/x$ and the x-axis from $x = 10$ to $x = 20$ is the same as the area between the curve and the x-axis from $x = 1$ to $x = 2$.

b. Show that the area between the curve $y = 1/x$ and the x-axis from ka to kb is the same as the area between the curve and the x-axis from $x = a$ to $x = b$ $(0 < a < b, k > 0)$.

Initial Value Problems

35. Show that $y = x^2 + \int_{1}^{x} \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2 y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

36. Show that $y = \int_{0}^{x} \left(1 + 2\sqrt{\sec t}\right) dt$ solves the initial value problem

$$\frac{d^2 y}{dx^2} = \sqrt{\sec x}\tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 37 and 38 in terms of integrals.

37. $\frac{dy}{dx} = \frac{\sin x}{x}, \quad y(5) = -3$

38. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Solve the initial value problems in Exercises 39–42.

39. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - x^2}}, \quad y(0) = 0$

40. $\dfrac{dy}{dx} = \dfrac{1}{x^2 + 1} - 1, \quad y(0) = 1$

41. $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2 - 1}}, \quad x > 1; \quad y(2) = \pi$

42. $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2} - \dfrac{2}{\sqrt{1 - x^2}}, \quad y(0) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 43–72.

43. $\int 2(\cos x)^{-1/2} \sin x \, dx$

44. $\int (\tan x)^{-3/2} \sec^2 x \, dx$

45. $\int (2\theta + 1 + 2\cos(2\theta + 1))\, d\theta$

46. $\int \left(\dfrac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi) \right) d\theta$

47. $\int \left(t - \dfrac{2}{t}\right)\left(t + \dfrac{2}{t}\right) dt$

48. $\int \dfrac{(t + 1)^2 - 1}{t^4}\, dt$

49. $\int \sqrt{t} \sin(2t^{3/2})\, dt$

50. $\int (\sec\theta \tan\theta)\sqrt{1 + \sec\theta}\, d\theta$

51. $\int e^x \sec^2(e^x - 7)\, dx$

52. $\int e^y \csc(e^y + 1)\cot(e^y + 1)\, dy$

53. $\int (\sec^2 x)\, e^{\tan x}\, dx$

54. $\int (\csc^2 x)\, e^{\cot x}\, dx$

55. $\int_{-1}^{1} \dfrac{dx}{3x - 4}$

56. $\int_{1}^{e} \dfrac{\sqrt{\ln x}}{x}\, dx$

57. $\int_{0}^{4} \dfrac{2t}{t^2 - 25}\, dt$

58. $\int \dfrac{\tan(\ln v)}{v}\, dv$

59. $\int \dfrac{(\ln x)^{-3}}{x}\, dx$

60. $\int \dfrac{1}{r}\csc^2(1 + \ln r)\, dr$

61. $\int x3^{x^2}\, dx$

62. $\int 2^{\tan x}\sec^2 x\, dx$

63. $\int \dfrac{3\, dr}{\sqrt{1 - 4(r - 1)^2}}$

64. $\int \dfrac{6\, dr}{\sqrt{4 - (r + 1)^2}}$

65. $\int \dfrac{dx}{2 + (x - 1)^2}$

66. $\int \dfrac{dx}{1 + (3x + 1)^2}$

67. $\int \dfrac{dx}{(2x - 1)\sqrt{(2x - 1)^2 - 4}}$

68. $\int \dfrac{dx}{(x + 3)\sqrt{(x + 3)^2 - 25}}$

69. $\int \dfrac{e^{\sin^{-1}\sqrt{x}}\, dx}{2\sqrt{x - x^2}}$

70. $\int \dfrac{\sqrt{\sin^{-1} x}\, dx}{\sqrt{1 - x^2}}$

71. $\int \dfrac{dy}{\sqrt{\tan^{-1} y}\,(1 + y^2)}$

72. $\int \dfrac{(\tan^{-1} x)^2\, dx}{1 + x^2}$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 73–112.

73. $\int_{-1}^{1} (3x^2 - 4x + 7)\, dx$

74. $\int_{0}^{1} (8s^3 - 12s^2 + 5)\, ds$

75. $\int_{1}^{2} \dfrac{4}{v^2}\, dv$

76. $\int_{1}^{27} x^{-4/3}\, dx$

77. $\int_{1}^{4} \dfrac{dt}{t\sqrt{t}}$

78. $\int_{1}^{4} \dfrac{(1 + \sqrt{u})^{1/2}}{\sqrt{u}}\, du$

79. $\int_{0}^{1} \dfrac{36\, dx}{(2x + 1)^3}$

80. $\int_{0}^{1} \dfrac{dr}{\sqrt[3]{(7 - 5r)^2}}$

81. $\int_{1/8}^{1} x^{-1/3}(1 - x^{2/3})^{3/2}\, dx$

82. $\int_{0}^{1/2} x^3(1 + 9x^4)^{-3/2}\, dx$

83. $\int_{0}^{\pi} \sin^2 5r\, dr$

84. $\int_{0}^{\pi/4} \cos^2\left(4t - \dfrac{\pi}{4}\right) dt$

85. $\int_{0}^{\pi/3} \sec^2\theta\, d\theta$

86. $\int_{\pi/4}^{3\pi/4} \csc^2 x\, dx$

87. $\int_{\pi}^{3\pi} \cot^2 \dfrac{x}{6}\, dx$

88. $\int_{0}^{\pi} \tan^2 \dfrac{\theta}{3}\, d\theta$

89. $\int_{-\pi/3}^{0} \sec x \tan x\, dx$

90. $\int_{\pi/4}^{3\pi/4} \csc z \cot z\, dz$

91. $\int_{0}^{\pi/2} 5(\sin x)^{3/2}\cos x\, dx$

92. $\int_{-\pi/2}^{\pi/2} 15\sin^4 3x \cos 3x\, dx$

93. $\int_{0}^{\pi/2} \dfrac{3\sin x \cos x}{\sqrt{1 + 3\sin^2 x}}\, dx$

94. $\int_{0}^{\pi/4} \dfrac{\sec^2 x}{(1 + 7\tan x)^{2/3}}\, dx$

95. $\int_{1}^{4} \left(\dfrac{x}{8} + \dfrac{1}{2x}\right) dx$

96. $\int_{1}^{8} \left(\dfrac{2}{3x} - \dfrac{8}{x^2}\right) dx$

97. $\int_{-2}^{-1} e^{-(x+1)}\, dx$

98. $\int_{-\ln 2}^{0} e^{2w}\, dw$

99. $\int_{0}^{\ln 5} e^r(3e^r + 1)^{-3/2}\, dr$

100. $\int_{0}^{\ln 9} e^\theta(e^\theta - 1)^{1/2}\, d\theta$

101. $\int_{1}^{e} \dfrac{1}{x}(1 + 7\ln x)^{-1/3}\, dx$

102. $\int_{1}^{3} \dfrac{(\ln(v + 1))^2}{v + 1}\, dv$

103. $\int_{1}^{8} \dfrac{\log_4\theta}{\theta}\, d\theta$

104. $\int_{1}^{e} \dfrac{8\ln 3 \log_3\theta}{\theta}\, d\theta$

105. $\int_{-3/4}^{3/4} \dfrac{6\, dx}{\sqrt{9 - 4x^2}}$

106. $\int_{-1/5}^{1/5} \dfrac{6\, dx}{\sqrt{4 - 25x^2}}$

107. $\int_{-2}^{2} \dfrac{3\, dt}{4 + 3t^2}$

108. $\int_{\sqrt{3}}^{3} \dfrac{dt}{3 + t^2}$

109. $\int_{1/\sqrt{3}}^{1} \dfrac{dy}{y\sqrt{4y^2 - 1}}$

110. $\int_{4\sqrt{2}}^{8} \dfrac{24\, dy}{y\sqrt{y^2 - 16}}$

111. $\int_{\sqrt{2}/3}^{2/3} \dfrac{dy}{|y|\sqrt{9y^2 - 1}}$

112. $\int_{-2/\sqrt{5}}^{-\sqrt{6}/\sqrt{5}} \dfrac{dy}{|y|\sqrt{5y^2 - 3}}$

Average Values

113. Find the average value of $f(x) = mx + b$
   a. over $[-1, 1]$ **b.** over $[-k, k]$

114. Find the average value of
   a. $y = \sqrt{3x}$ over $[0, 3]$ **b.** $y = \sqrt{ax}$ over $[0, a]$

115. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

116. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

117. **a.** Verify that $\int \ln x\, dx = x \ln x - x + C$.

b. Find the average value of $\ln x$ over $[1, e]$.

118. Find the average value of $f(x) = 1/x$ on $[1, 2]$.

T **119.** Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. (See Exercise 98, Section 3.6.) This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

T **120. Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of one mole (gram molecule) of a gas with constant volume by 1°C. The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°\text{C}$ and the temperature at which it is attained.

Differentiating Integrals

In Exercises 121–128, find dy/dx.

121. $y = \int_2^x \sqrt{2 + \cos^3 t}\, dt$

122. $y = \int_2^{7x^2} \sqrt{2 + \cos^3 t}\, dt$

123. $y = \int_x^1 \frac{6}{3 + t^4}\, dt$

124. $y = \int_{\sec x}^2 \frac{1}{t^2 + 1}\, dt$

125. $y = \int_{\ln x^2}^0 e^{\cos t}\, dt$

126. $y = \int_1^{e^{\sqrt{x}}} \ln(t^2 + 1)\, dt$

127. $y = \int_0^{\sin^{-1} x} \frac{dt}{\sqrt{1 - 2t^2}}$

128. $y = \int_{\tan^{-1} x}^{\pi/4} e^{\sqrt{t}}\, dt$

Theory and Examples

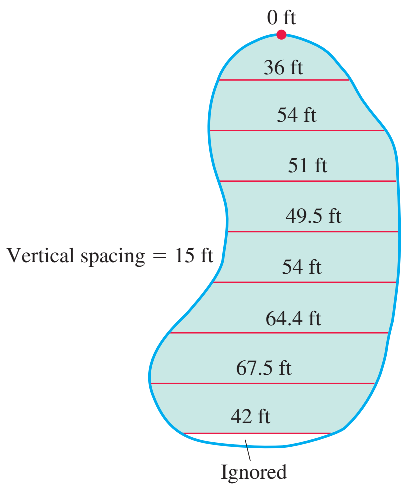
129. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

130. Suppose that $f(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F and give a reason for your answer.

131. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\, dt$. Explain the main steps in your calculation.

132. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2))\, dt$. Explain the main steps in your calculation.

133. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for $10,000. The cost to clear the land will be $0.10 a square foot, and the lot will cost $2.00 a square foot to pave. Can the job be done for $10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



134. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening his parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

a. At what altitude does A's parachute open?

b. At what altitude does B's parachute open?

c. Which skydiver lands first?