COMBINATORICS AND PROBABILITY

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Three questions.

Probability theory started in 1654 with a letter by an aristocrat and a notorious gambler Chevalier de Méré to Blaise Pascal. In essence, the letter said: "I used to bet even money that I would get at least one 6 in four rolls of a fair die. The probability of this is 4 times the probability of getting a 6 on a single die, i.e. 2/3; clearly I had an advantage and indeed I was making money. Now I bet even money that within 24 rolls of two dice I would get at least one 12. This still has the same advantage, but I'm losing money. Why?"

In a family of 4 children, what is the probability of 2 boys? Since there are 5 possible outcomes $(0, \ldots, 4$ boys), shouldn't it be 1/5?

Is the ranking of poker hands based on their probabilities?

All our examples will feature *fair* coins and dice, as well as *shuffled* decks of cards.

Equally likely outcomes.

Suppose an experiment with n possible outcomes is performed. *Assume that these are all equally likely.* If an event E comprises m different outcomes (these outcomes are *good* for E, others are *bad*), then its probability of E equals

$$P(E) = \frac{m}{n}.$$

Example. Toss a fair coin twice. What are the equally likely outcomes?

Answer: HH, HT, TH, TT.

Hence the probability of exactly one H, the event consisting of 2 outcomes, is $1/2$.

Example. Roll two dice. What is the most likely sum?

There are 36 outcomes, and the sum could be 2 (1 outcome), 3 (2 outcomes), 4 (3), 5 (4), 6 (5), 7 (6), 8(5), 9 (4), 10 (3), 11 (2), or 12 (1).

Counting.

The basic principle: if an experiment consists of two parts, the first part has m outcomes, and the second part has n outcomes regardless what happens at the first part, then the total number of outcomes is $m \cdot n$.

Example. Solution of de Méré's problem. In the first game, the number of outcomes is 6^4. and the number of bad outcomes for winning is 5^4, so

$$P(\text{win}) = 1 - (5/6)^4 \approx 0.5177.$$

The mistake was in counting $4 \cdot 6^3$ good outcomes, but this *overcounts* since many good outcomes have more than one 6 and are hence counted more than once. In the second game, the number of outcomes is 36^{24} and the number of bad ones 35^{24}, giving

$$P(\text{win}) = 1 - (35/36)^{24} \approx 0.4914.$$

Example. Roll the die 4 times. What's the probability that you get different numbers? The number of good outcomes is $6 \cdot 5 \cdot 4 \cdot 3$, and the probability $5/18 \approx 0.278$.

Permutations.

If you have n objects, there are $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ ways to order them all, that is, fill n ordered slots with them. The number of ways to fill a smaller number $k \leq n$ of ordered slots is $n!/k! = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$.

Example. Take the 4 Q's and 4 K's, shuffle the 8 cards, then lay them down on the table from left to right. What is the probability that the $K\heartsuit$ and $Q\heartsuit$ end up next to each other?

Answer: $2 \cdot 7!/8! = 0.25$.

Example. What is the probability that, in a shuffled deck of cards, all cards of the same suits are together?

The answer is $4! \cdot (13!)^4/52! \approx 4.5 \cdot 10^{-28}$. Such an event never happens in practice.

What is the probability that just the hearts are together?

Answer: $13!40!/52! \approx 6 \cdot 10^{-11}$.

Birthday problem.

Assume that k people are in the room, and that every day of the year is equally likely to be the birthday of any of the people.

$$P(\text{at least two people share a birthday})$$
$$= 1 - \frac{365 \cdot 364 \ldots (365 - k + 1)}{365^k}.$$

This is 0.507 for $k = 23$, and 0.891 for $k = 40$.

Combinations.

Example. Shuffle the 13 hearts, and deal yourself the top three of them. What is the probability you will receive exactly the 3 face cards (J, Q, K)?

We already have a way to solve this problem. The number of all outcomes is 13!, the number of good outcomes is $3! \cdot 10!$, and the probability about 0.0035.

Another way to think about this is to redefine an outcome to be the set of cards you may receive (the order in which you receive them, obviously, does not matter). By a little leap of faith, *all these outcomes are equally likely*. In this case exactly one outcome is good, and so the number of outcomes is

$$\frac{13!}{10! \cdot 3!} = \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} = \binom{13}{3}$$

The number

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

is called "n choose k" or a *binomial symbol*. It represents the number of ways to make an unordered selection of k objects from the set of n objects. (We *define* $0! = 1$ and $\binom{n}{0} = 1$.

Two famous occurencies of binomial symbols.

Binomial theorem:

$$(x+y)^n =$$
$$\binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \cdots + \binom{n}{k}x^ky^{n-k} + \cdots + \binom{n}{n}x^ny^0.$$

Pascal's Triangle: Start with $\ldots 0001000 \ldots$ Form the next line by adding each two successive numbers on the previous line, then repeat.

Numbers of heads in coin tosses.

Example. In 4 coin tosses, what are the probabilities of different numbers of heads?

Of course, there are $2^4 = 16$ different outcomes, and one could list all of them to see which are good. Another way is to observe that, e.g., the number of outcomes with 2 heads is exactly $\binom{4}{2} = 6$, the choice representing tosses on which the heads occur. So,

$$\begin{aligned} &P(0 \text{ heads}) = P(4 \text{ heads}) = 1/16, \\ &P(1 \text{ heads}) = P(3 \text{ heads}) = 1/4, \\ &P(2 \text{ heads}) = 3/8. \end{aligned}$$

In a 4 child family, the $3:1$ gender imbalance is more likely that the $2:2$ balance!

Example. Probability of exactly 5 heads in 10 coin tosses is

$$\binom{10}{5} 2^{-10} = 63 \cdot 2^{-8} = 0.246.$$

Two more examples.

Example. Deal 13 cards from a shuffled deck to each of the four players. What's the probability that each receives 1 ace?

There are two ways to think about this problem. In the first, the outcomes are all orders of cards, 52! of them. The number of good events is then $13^4 \cdot 4! \cdot 48!$.

In the second, you imagine cards put at random into 52 slots, first 13 correspond to the 1st player, etc. The aces, of course, occupy four random slots, each selection of the four slots being equally likely. The answer then is

$$13^4/\binom{52}{4} \approx 0.1055.$$

Example. Roll a die 12 times. What is the probability that each number appears exactly twice?

Answer:

$$\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}6^{-12} \approx 0.0034.$$

Poker hands.

In the definitions, the word *value* refers to A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2. This sequence also describes the relative rankings of cards, with one exception: an Ace may be regarded as 1 for the purposes of making a straight:

(a) *one pair*: two cards of same value plus 3 cards with different values

$$J\spadesuit\ J\clubsuit\ 9\heartsuit\ Q\clubsuit\ 3\spadesuit$$

(b) *two pairs*: two pairs plus another card of different value

$$J\spadesuit\ J\clubsuit\ 9\heartsuit\ 9\clubsuit\ 3\spadesuit$$

(c) *three of a kind*: three cards of the same value plus two with different values

$$J\spadesuit\ J\clubsuit\ J\heartsuit\ 9\clubsuit\ 3\spadesuit$$

(d) *straight*: five cards with consecutive values

$$5\heartsuit\ 4\clubsuit\ 3\clubsuit\ 2\heartsuit\ A\spadesuit$$

(e) *flush*: five cards of the same suit

$$K\clubsuit\ 9\clubsuit\ 7\clubsuit\ 6\clubsuit\ 3\clubsuit$$

(f) *full house*: a three of a kind and a pair

$$J\clubsuit\ J\diamondsuit\ J\heartsuit\ 9\clubsuit\ 9\spadesuit$$

(g) *four of a kind*: four cards of the same value

$$J\clubsuit\ J\diamondsuit\ J\heartsuit\ J\clubsuit\ 9\spadesuit$$

(e) *straight flush*: five cards of the same suit with consecutive values

$$A\clubsuit\ K\clubsuit\ Q\clubsuit\ J\clubsuit\ 10\clubsuit$$

hand	probability
one pair	0.422569
two pairs	0.047539
three of a kind	0.021128
straight	0.003940
flush	0.001981
full house	0.001441
four of a kind	0.000240
straight flush	0.000015

Note. The probability of straight above includes the possibility of a straight flush. Subtract the two to get the probability of a straight which is not a straight flush.

How does one compute these?

The number of outcomes is $\binom{52}{5} = 2,598,960$. For the *three of a kind*, the number of good outcomes is obtained by choosing the value for the three cards with the same value, then the values of other two cards, then three cards from the four of the same chosen value, then a card from each of the two remaining chosen values:

$$13 \cdot \binom{12}{2} \cdot \binom{4}{3} \cdot 4^2.$$

For *two pairs*, the number of good outcomes is

$$\binom{13}{2} \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4.$$

Guessing hats' colors.

Three people are in the room, they each wear a hat, either red or blue. Each can see the others' hats, but not his or her own. With no communication, the players must simultaneously either guess their color, or pass. The trio wins if at least one person guesses correctly, and none guesses incorrectly, otherwise they lose. Can they agree beforehand on a strategy that will allow them to win with probability more than 0.5?

Here is the crucial additional piece of information for them: the three colors are chosen by three flips of a fair coin.

Note: if one of them is John, and he always guesses red while the other two always pass, then $P(\text{win})=0.5$.

A much better strategy is as follows. Each player who sees different colors (of the two other hats) passes, while a player who sees same colors guesses the opposite color.

This way, the trio wins exactly when all hats are not of equal color, which happens with probability 3/4!

A non–transitive card game.

Three stacks (**A**, **B**, and **C**), contain three cards each. Two players are each assigned a stack. Each chooses a random card from his own stack; the highest card wins.

- Stack A card values: 1, 6, 8.
- Stack B card values: 3, 5, 7.
- Stack C card values: 2, 4, 9

$P(\mathbf{A}$ beats $\mathbf{B})=5/9$,

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