Math 280: Quantum Probability
Homework 1

This problem set is due Tuesday, October 23, by 8pm in my mailbox.

1.1. The point of this exercise is to illustrate the *probabilistic method*, in which probabilistic reasoning establishes theorems that have no statement of randomness in them. Prove that there exists a graph Γ with 42 vertices such that neither Γ nor its complement has a complete subgraph with 8 vertices. In other words, if you are familiar with a Ramsey numbers, $R(8) \geq 43$. (Hint: Of course you should see what happens if you choose Γ at random.)

1.2. Recall the notation from class that M_d is the algebra of $d \times d$ complex matrices.

(a) I said that $(M_d)_+ = \{y^*y\}$ is the cone of positive, semi-definite Hermitian matrices. Prove this fact.

(b) In general if K is a convex cone, then one talks about its extreme rays rather than its extreme points, since the apex of K is its only extreme point. Show that the points in the extreme rays of $(M_d)_+$ are exactly rank 1 Hermitian matrices, that thus factor as $x = |\psi\rangle\langle\psi|$.

(c) Use part (b) to show that $M_d^+ = (M_d)_+$ if we use the trace formula $\rho(x) = tr(\rho x)$ to identify M_d with its dual vector space.

(d) Use (b) and (c) to find the extreme points of the convex body M_d^Δ.

1.3. Prove the statement from class that the state region M_2^Δ is a round unit ball in the coordinates $(a,b,c) \in \mathbb{R}^3$, where the general state ρ is expressed as

$$\rho = \frac{I + a\sigma_x + b\sigma_y + c\sigma_z}{2}$$

and each $\sigma_\cdot$ is a Pauli matrix. (Hint: Take the determinant.)

1.4. Recall the definition from class that if X is a Banach space, then a *pre-dual* $X^\#$ of X is a subspace $X^\# \subseteq X^*$ such that the evaluation map induces an isometry $(X^\#)^* \to X$.

(a) Let $c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$ be the Banach space of sequences that converge to 0 (with the sup norm). Show that $c_0(\mathbb{N})$ is a predual of $\ell^1(\mathbb{N})$. (Hint: To get started, you should recognize that the elementary vectors with a 1 in one position and 0 everywhere else are a topological basis for $c_0(\mathbb{N})$. Hence every functional $\phi \in c_0(\mathbb{N})^*$ is determined by its values on this basis and is described by a sequence of numbers of some kind.)

***(b)** Let $c_1(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$ be the Banach space of sequences $(a_1, a_2, a_3, \ldots)$ that converge to the first term a_1. Show that $c_1(\mathbb{N})$ is also a predual of $\ell^1(\mathbb{N})$.

(c) Show that if a Banach space X has a predual, then its unit ball has a non-empty set of extreme points. (Hint: Look up the statements of the Banach-Alaoglu theorem and the Krein-Milman theorem.)

***(d)** Show that $c_0(\mathbb{N})$ does not have a predual.

***1.5.** Let K be a convex set in some real vector space V and let $p \in K$. Then p is *k-extreme* when k is the largest integer such that $p \pm \varepsilon v \in K$ for some $\varepsilon > 0$ and some k linearly independent vectors $v \in V$. Prove that for every $1 \leq k \leq d$, the qudit state region M_d^Δ has a $(2dk - k^2 - 1)$-dimensional manifold of $(k^2 - 1)$-extreme points, and doesn't have any j-extreme points for any other j. For instance M_3^Δ has a 4-manifold of 0-extreme points, a 7-manifold of 3-extreme points, and an 8-manifold of 8-extreme points.