Math 280: Quantum Probability
Homework 2

This problem set is due Tuesday, November 6, by 8pm in my mailbox.

2.1. In class I mentioned that a finite-dimensional von Neumann algebra is a direct sum of matrix algebras, and I have discussed the state regions $(d\mathbb{C})^\Delta$ and M_d^Δ. What is the state region $(M_1 \oplus M_2)^\Delta$ of the 5-dimensional von Neumann algebra $M_1 \oplus M_2$? (Hint: It's a cone, the kind with a base and an apex. What is the base and what is the apex?)

2.2. Obtaining hidden measurements by averaging unitaries. In this exercise, let $\mathscr{M}$ be any von Neumann algebra.

(a) Confirm that if $u \in \mathscr{M}_{S^1}$ is unitary, *i.e.*, if $u^*u = u^*u = 1$, then the map $x \mapsto uxu^*$ is an automorphism of $\mathscr{M}$ (by definition an *inner* automorphism), while the map $E_u(\rho)(x) = \rho(uxu^*)$ is a continuous bijection on the space of states $\mathscr{M}^\Delta$.

(b) Let $b \in \mathscr{M}_{\mathbb{Z}/2}$ be boolean and let $\bar{b} = 1 - b$ be its complement. Confirm what I said in class, that each of $u_\pm = b \pm \bar{b}$ is unitary and that the hidden measurement map

$$\rho_{(b)} = \rho(bxb) + \rho(\bar{b}x\bar{b})$$

can also be expressed as

$$\rho_{(b)} = \frac{E_{u_+}(\rho) + E_{u_-}(\rho)}{2}.$$

(Thus, secretly measuring a boolean of a physical system has the same effect as choosing randomly between two certain reversible evolutions.)

(c) Show that $u_\theta = b + e^{i\theta}\bar{b}$ is unitary for every real θ, and show that

$$\rho_{(b)} = \frac{1}{2\pi}\int_0^{2\pi} E_{u_\theta}(\rho)\,d\theta.$$

***(d)** Generalize parts (b) and (c) to a hidden measurement taking values in a finite set. *I.e.*, recover the hidden measurement transformation as a larger sum of unitary evolutions, or likewise a bigger integral.

2.3. Particle oscillations. Suppose that a quantum system $\mathscr{M} = \mathscr{L}(\mathscr{H})$ is given by a Hilbert space $\mathscr{H}$. In class I mentioned that I mentioned that a vector state of $\mathscr{M}$ typically evolves in time by a Schrödinger equation

$$-i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle.$$

The operator $H \in \mathscr{L}(\mathscr{H})_\mathbb{R}$ on the right side is called the *Hamiltonian*, and as a measurement it is called *energy* or *mass*. (By special relativity, energy can always be interpreted as mass. Meanwhile Planck's constant $\hbar$ on the left side is a fundamental conversion factor that evidently has units of time-energy. You can check that this is the same units as angular momentum.)

(a) Suppose that a particle P comes in two "mass flavors" A and B, each with their own mass m_A and m_B. So, the particle has a 2-dimensional flavor Hilbert space $\mathscr{H} \cong \mathbb{C}^2$ with an orthonormal basis $|A\rangle$ and $|B\rangle$, and its flavor state evolves over time using the Hamiltonian

$$H = \begin{pmatrix} m_A & 0 \\ 0 & m_B \end{pmatrix}.$$

Suppose that P also has two "collision flavors" C and D, representing the states in which P is created or detected in experiments. These collision flavors are another basis of $\mathscr{H}$ which could have a change of basis matrix

$$|C\rangle = (\cos\theta)|A\rangle + (\sin\theta)|B\rangle$$
$$|D\rangle = -(\sin\theta)|A\rangle + (\cos\theta)|B\rangle.$$

Suppose that P is created in state $|C\rangle$ and, after it lives for time t, is measured (and eaten up) by a detector in either state $|C\rangle$ or $|D\rangle$. What is the probability that it will be seen in state $|C\rangle$?

(b) The matrix in part (a) does not look fully general; you would expect it to be possibly any 2×2 unitary matrix U. Remember that as descriptions of states of a quantum system, the vectors $|A\rangle$, $|B\rangle$, $|C\rangle$, and $|D\rangle$ are only each well-defined up to a global phase. Show that every unitary 2×2 matrix can be converted to a rotation matrix by changing the phases of its rows and columns. (Hint: Write the entries of U in polar form,

$$U = \begin{pmatrix} r_1 e^{i\theta_1} & r_2 e^{i\theta_1} \\ r_3 e^{i\theta_3} & r_4 e^{i\theta_4} \end{pmatrix}.$$

Given that U is unitary, what are the constraints on the lengths and angles of the entries?)

***(c)** Although the actual number of flavors is three rather than two, the phenomenon of two competing flavor bases occurs in real life for electrons, neutrinos, and quarks. Each of these three types of particles has a mass basis and a collision basis, where the type of collision is via a force called the *weak interaction*. Using the Internet, determine whether the 3 electron flavors are named in the mass basis or the weak interaction basis, and likewise neutrinos and quarks. (Actually quarks have 6 flavors, but they come in 3 pairs.)

2.4. Suppose that Alice and Bob have (or are!) quantum systems with finite-dimensional Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ with some chosen orthonormal bases. Then a joint pure state $|\psi_{AB}\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$ is described by a complex matrix Ψ_{jk}, where the index j corresponds to $\mathscr{H}_A$ and the index k corresponds to $\mathscr{H}_B$. Show that the matrices of the marginal states ρ_A and ρ_B are given by the matrix products

$$\rho_A = \Psi\Psi^* \qquad \rho_B = \Psi^*\Psi.$$

2.5. Borrowing from the previous problem, suppose that $\mathscr{H}_A$ is d-dimensional and has basis $|1\rangle, |2\rangle, \ldots, |d\rangle$, while $\mathscr{H}_B$ is 2-dimensional and has basis $|\text{yes}\rangle$ and $|\text{no}\rangle$. Let $S \subseteq \{1,2,3,\ldots,d\}$ be a subset of the integers from 1 to d, and let $b \in \mathscr{L}(\mathscr{H}_A)$ be the boolean that asks Alice, "Is your state in the set S"?

(a) Confirm that the operator

$$u = b \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \bar{b} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is unitary and is in fact a permutation matrix.

(b) Suppose that Alice initially has the state ρ_A (pure or mixed) and Bob initially has the state $|\text{no}\rangle$, and we apply the operator u from part (a) to get a new state ρ'_{AB}. Argue that Bob asked Alice b, in the sense that

$$\Pr_{\rho'_{AB}}[\text{yes}] = \Pr_\rho[b] = \rho(b) \qquad \Pr_{\rho'_{AB}}[\text{no}] = \Pr_\rho[\bar{b}] = \rho(\bar{b}).$$

(c) Following part (b), suppose that we begin in the state $\rho_A \otimes |\text{no}\rangle\langle\text{no}|$, then apply u, and then Bob departs leaving Alice with a new marginal state ρ'_A. Show that this is the hidden measurement state

$$\rho'_A = (\rho_A)_{(b)} = b\rho_A b + \bar{b}\rho_A\bar{b}.$$