Math 280: Quantum Probability
Homework 3

This problem set is due Tuesday, November 27, by 8pm in my mailbox.

3.1. Recall from class and homework 1 that the state region M_2^Δ is a round ball. Consider a linear map

$$E : M_2^\# \to M_2^\#$$

that preserves the uniform state in the center of $M_2^\Delta \subseteq M_2^\#$ and rescales M_2^Δ by a factor of $t \in \mathbb{R}$. The condition that E is TPP says that $E(M_2^\Delta) \subseteq M_2^\Delta$, which thus means that $t \in [-1,1]$. In order for E to be a quantum map, it needs to be TPCP, not just TPP. Show that E is TPCP when $t \in [-\frac{1}{3}, 1]$.

To do this problem, you can use without proof a simplification of the CP condition. The full condition is that a linear map $E : M_2^\# \to \mathscr{B}^\#$ is completely positive when

$$E \otimes I : (M_2 \otimes \mathscr{C})^\# \to (\mathscr{B} \otimes \mathscr{C})^\#$$

is positive for all $\mathscr{C}$. It suffices to let $\mathscr{C} = M_2$ and check that $E(\rho_{AB}) \in (M_2 \otimes \mathscr{C})^+$, where

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| \qquad \psi_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

is a Bell state.

3.2. In this question we consider a quantum map $F = D \circ U \circ E$ which is a composition as follows:

$$(d\mathbb{C})^\Delta \xrightarrow{E} M_d^\Delta \xrightarrow{U} M_d^\Delta \xrightarrow{D} (d\mathbb{C})^\Delta.$$

Let $\{[1],[2],\ldots,[d]\}$ be the deterministic states of $(d\mathbb{C})^\Delta$ and let $|1\rangle, |2\rangle, \ldots, |d\rangle$ be an orthonormal basis of states of the Hilbert space $\mathscr{H} = \mathbb{C}^d$ on which M_d acts. We let

$$E([k]) = |k\rangle\langle k| \qquad U(\rho) = u\rho u^* \qquad D(\rho) = \sum_k \langle k|\rho|k\rangle [k],$$

where u is some unitary matrix, and D is the indicated measurement in the standard basis. So the interpretation of the composition F is that we start with a classical d-state digit, encode it in a qudit, apply a unitary, and then decode back to a classical digit by measuring the qudit.

(a) Confirm that F is a doubly stochastic matrix given by $F_{jk} = |u_{jk}|^2$.

(b) Show that if $d = 2$, then every doubly stochastic matrix F is induced by some unitary matrix u.

*(c) Show that if $d = 3$, then not every doubly stochastic matrix F comes from a unitary matrix u in this manner.

3.3. The (simplest) no-cloning theorem: Prove that if $d \geq 2$, then there does not exist a quantum map

$$E : M_d^\Delta \to (M_d \otimes M_d)^\Delta$$

such that

$$E(|\psi\rangle\langle\psi|) = |\psi,\psi\rangle\langle\psi,\psi| \qquad |\psi,\psi\rangle = |\psi\rangle \otimes |\psi\rangle$$

for every pure state $|\psi\rangle \in \mathbb{C}^d$. (Hint: E can't even be linear. Think about $d = 2$ first.)

***3.4.** Consider a composition of quantum maps $F = D \circ E$ of the form

$$(3\mathbb{C})^\Delta \xrightarrow{E} M_2^\Delta \xrightarrow{D} (3\mathbb{C})^\Delta.$$

So E encodes a classical trit into a qubit and D decodes it back again. TPP implies TPCP for both D and E, so they are simply any affine-linear maps between the triangle $(3\mathbb{C})^\Delta$ and the round ball M_2^Δ. Show that if one of the three configurations $[k]$ of $(3\mathbb{C})^\Delta$ is chosen at random, then the average probability that $F([k])$ is in state $[k]$ is at most $2/3$. (Interpretation: You cannot encode a trit into a qubit any more reliably than you can encode a trit into a bit. The pigeonhole principle generalizes to this case.)

3.5. Let $\mathscr{H} = \mathbb{C}^d$ be the standard d-dimensional Hilbert space with basis $|1\rangle, |2\rangle, \ldots, |d\rangle$, and suppose that it is houses two particles whose state is in the bosonic Hilbert space is $S^2(\mathscr{H}) \subseteq \mathscr{H} \otimes \mathscr{H}$. To be more explicit, the Hilbert space $S^2(\mathscr{H})$ has a basis

$$|k\rangle \otimes |k\rangle \qquad \text{and} \qquad \frac{|j\rangle \otimes |k\rangle + |k\rangle \otimes |j\rangle}{\sqrt{2}}, j < k.$$

(a) Suppose that the two particles are given the uniform bosonic state ρ_{unif} in $\mathscr{L}(S^2(\mathscr{H}))^\Delta$. Suppose that we measure the first particle in the standard basis and it is found to be in state $|1\rangle$. (The measurement as describes breaks symmetry between the two particles, so they might not be bosonic afterwards.) What is the probability that the second particle, if also measured, is in any given state $|k\rangle$? (Warning: The answer depends on both k and d. Hint: Recall that the uniform state on $\mathbb{C}^d$ is given by $\rho_{\text{unif}} = \sum_j \frac{|k\rangle\langle k|}{d}$.)

***(b)** Generalize part (a) to n particles whose bosonic Hilbert space is $S^n(\mathbb{C}^d)$. If the n particles are in the uniform bosonic state and if the first ℓ are measured to be states $|k_1\rangle, |k_2\rangle, \ldots, |k_\ell\rangle$, then what is the probability distribution for the measurement of the $\ell + 1$st particle?