Extra Problems
Math 128A, Fall 2019

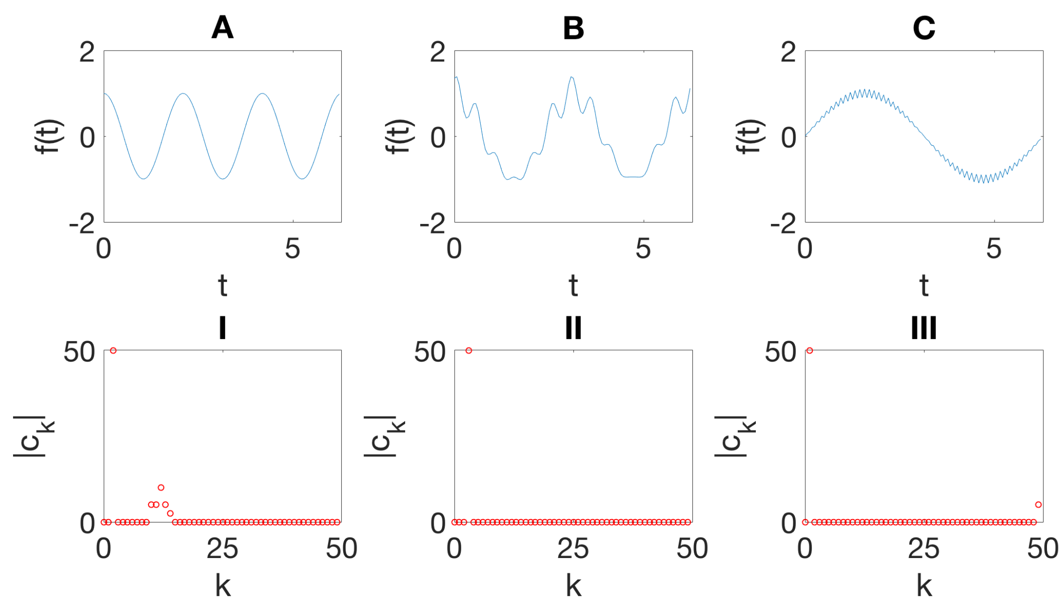
- Below are some problems to help study for the final exam. This is **not** a practice exam, but working these problems will help you study for the exam. You should additionally study your homework, textbook, notes from class, and additional problems given for the midterm practice.

- These problems focus only on the material after the midterm. The final exam is comprehensive. See the extra problems given before the midterm for reviewing the material from the first half of the class.

- For the final exam, you may bring one handwritten $8.5'' \times 11''$ sheet of notes to the exam.

1. Given a set of data points (x_k, y_k) for $k = 0 \ldots n$, find the line through the origin which best approximates the data in the least squares sense.

2. In the first row (plots A, B, and C) are functions $f(t)$ on the domain $t \in [0, 2\pi)$. In the second row (plots I, II, and III) are the absolute values of the discrete Fourier coefficients $|c_k|$ of each of the functions $f(t)$ computed with the FFT. Match each function plot to its Fourier coefficient plot.



3. Indicate which of the following statements are true and which are false, and provide an explanation for your answer.

   (a) If p is a monic polynomial of degree n, then

$$\max_{-1 \le x \le 1} |p(x)| \ge 2^{-n+1}$$

(b) Given a smooth function $f(x)$ and let $p_N(x)$ denote the polynomial which interpolates $f(x)$ at the $(N+1)$ equispaced nodes

$$-1+\frac{2j}{N}, \quad j=0,\cdots,N$$

then $\max_{-1\le x\le 1}|f(x)-p_N(x)| \to 0$ as $N\to\infty$.

(c) Simpson rule can be obtained by integrating a *quadratic* Lagrange interpolating polynomial, i.e. of degree 2. Therefore, the degree of precision for Simpson rule is 2.

(d) For an n-point quadrature, the maximum possible degree of precision possible is is $n+1$.

4. Given three orthonormal polynomials $\{P_0, P_1, P_2\}$

$$P_0(x)=1, \quad P_1(x)=\sqrt{12}(x-\frac{1}{2}), \quad P_2(x)=\sqrt{180}(x^2-x+\frac{1}{6})$$

with respect to a certain weight function $w(x)$ on the interval $[0,1]$, i.e.

$$\int_0^1 P_i(x)P_j(x)w(x)\,dx = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

(a) What is the maximum possible degree of precision for the quadrature rule below?

$$\int_0^1 f(x)w(x)\,dx \approx f(x_0)w_0 + f(x_1)w_1$$

Explain.

(b) Find the nodes x_0, x_1 and the weights w_0, w_1 that give the maximum precision.

5. Find the quadratic polynomial, q_2, that best approximates $f(x)=\sin(x)$ on $[0,\pi]$ by minimizing

$$\int_0^\pi (f(x)-q_2(x))^2\,dx,$$

using the Legendre polynomials. Some useful integrals

$$\int_0^\pi x\sin(x)\,dx=\pi, \qquad \int_0^\pi x^2\sin(x)\,dx=\pi^2-4.$$

6. (a) Find the quadratic polynomial that interpolates $f(x)=\sin(x)$ on $[0,\pi]$ at the points which give lowest maximum interpolation error.

(b) Give a bound on the maximum interpolation error.

7. Suppose you want to approximate the integral below.

$$\int_0^2 \cosh(x)\,dx$$

(a) Using the trapezoidal rule with equally spaced points, how many points are required to guarantee the error is below 10^6?

(b) Using Simpson's rule with equally spaced points, how many points are required to guarantee the error is below 10^6?

8. Derive Simpson's rule.

9. Suppose f is a continuously differentiable 2π-periodic function and can be represented by its Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp(ikx).$$

Let $S_n(x)$ be the degree n trigonometric polynomial

$$S_n(x) = \sum_{k=-n}^{n} d_k \exp(ikx).$$

Find the degree n trigonometric polynomial that minimizes

$$\int_{-\pi}^{pi} (f(x) - S_n(x))^2 \, dx.$$

Justify your answer.