Homework 3
Math 128A
Due Monday, 11/10/08, 11:00 a.m.

1. Suppose that $f \in C^4[a,b]$, and let S be the piecewise Hermite cubic polynomial that interpolates f and f' at the nodes $a = x_0 < \ldots < x_n = b$, then for $x \in [a,b]$

$$|f(x) - S(x)| \leq \frac{1}{384} \max_{x \in [a,b]} |f^{(4)}(x)| \max_j (x_{j+1} - x_j)^4.$$

   Derive this error bound using Theorem 3.9 from the textbook.

2. There are many different formulas to approximate the first derivative of a function. Three are listed below.

$$\text{(i)} \quad \frac{f(x+h) - f(x)}{h} \approx f'(x)$$

$$\text{(ii)} \quad \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} \approx f'(x)$$

$$\text{(iii)} \quad \frac{-f(x+2h) + 6f(x+h) - 3f(x) - 2f(x-h)}{6h} \approx f'(x)$$

   (a) Derive an expression for the leading order error for each of the above approximations. Determine the order of accuracy of each approximation.

   (b) Apply the difference formulas to $f(x) = \exp(-x)$ at $x = 0$ to approximate $f'(0)$. Use increasingly smaller values of h, and make a table of the approximate derivatives and a table of the absolute errors of each of the approximations.

   (c) Plot the absolute errors vs. h on a log-log graph.

   (d) Explain how the table of errors and the graph of the errors demonstrate the order of convergence.

3. The three point open Newton-Cotes formula is

$$\int_a^b f(x)\,dx \approx \frac{4h}{3}\Big(2f(x_1) - f(x_2) + 2f(x_3)\Big),$$

   where $h = (b-a)/4$ and $x_j = jh + a$.

   (a) Derive this formula by integrating the appropriate interpolating polynomial.

   (b) Apply the formula to the monomials x^k for $k = 0, 1, \ldots$ for $a = 0$ and $b = 1$ to determine the degree of precision.

   (c) Derive the integration formula based on the unequally spaced points $x_1 = a + h$, $x_2 = a + 2h$, $x_3 = a + 7h/2$, and determine its degree of precision.

   (d) Using a mathematical argument, explain the origin of the difference in precision between these two integration formulas.