

Homework 3
Math 128A

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1. Hermite cubic polynomial

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Theorem 3.9 states that the error for a Hermitian cubic polynomial is found by

$$f(x) = H_3(x) + \frac{(x - x_j)^2(x - x_{j+1})^2}{4!} f^{(4)}(\xi(x)).$$

Letting $S(x) = H_3(x)$, the error is bounded by

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$$|f(x) - S(x)| \leq \left| \frac{(x - x_j)^2(x - x_{j+1})^2}{4!} f^{(4)}(\xi(x)) \right|.$$

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Let $g(x) = (x - x_j)^2(x - x_{j+1})^2 = [(x - x_j)(x - x_{j+1})]^2 = (x^2 - (x_{j+1} + x_j)x + x_{j+1}x_j)^2$. Then solving $g'(x) = 0$ gives a bound for $g(x)$.

$$g'(x) = 2(x^2 - (x_{j+1} + x_j)x + x_{j+1}x_j)(2x - (x_{j+1} + x_j)) = 0.$$

The values of x for which this is true are $x_j, x_{j+1}, (x_{j+1} + x_j)/2$.

x	$g(x)$
x_j	0
$(x_{j+1} + x_j)/2$	$[(x_{j+1} - x_j)]^4/16$
x_{j+1}	0

Therefore, $|g(x)| \leq [(x_{j+1} - x_j)/2]^4$, where j is chosen to give us a maximum difference between 2 successive nodes. Additionally, ξ is generally unknown, but we can bound the term $f^{(4)}(\xi(x))$ by the maximum value of $|f^{(4)}(x)|$ on an interval $[a, b]$. Thus, we have

$$|f(x) - S(x)| \leq \frac{1}{384} \max_{x \in [a, b]} |f^{(4)}(x)| \max_j (x_{j+1} - x_j)^4.$$

2. Approximations of $f'(x)$

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(a) Know that

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) + \dots \quad (1)$$

$$f(x + 2h) = f(x) + 2hf'(x) + 2h^2 f''(x) + \frac{4h^3}{3} f'''(x) + \frac{2h^4}{3} f^{(4)}(x) + \dots \quad (2)$$

$$f(x - h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \frac{h^4}{24} f^{(4)}(x) - \dots \quad (3)$$

and let f_j refer to $f(x + jh)$ for brevity. Then,

$$\begin{aligned}
 \text{i. } \frac{f_1 - f_0}{h} &= \frac{1}{h} \left(hf'(x) + \frac{h^2}{2} f''(x) + \dots \right) \\
 &= f'(x) + \frac{h}{2} f''(x) + \dots \\
 &= f'(x) + O(h)
 \end{aligned}$$

This approximation is of linear accuracy.

$$\begin{aligned}
 \text{ii. } \frac{-f_2 + 4f_1 - 3f_0}{2h} &= \frac{1}{2h} \left(hf'(x)(-2 + 4) + \frac{h^3}{6} f'''(x)(8 - 4) + \dots \right) \\
 &= f'(x) - \frac{h^2}{3} f'''(x) + \dots \\
 &= f'(x) + O(h^2)
 \end{aligned}$$

This approximation has second ~~degree~~ ^{order} accuracy.

$$\begin{aligned}
 \text{iii. } \frac{-f_2 + 6f_1 - 3f_0 - 2f_{-1}}{6h} &= \frac{1}{6h} \left(hf'(x)(-2 + 6 + 2) + \frac{h^4}{24} f^{(4)}(-16 + 6 + 2) + \dots \right) \\
 &= f'(x) - \frac{h^3}{12} f'''(x) + \dots \\
 &= f'(x) + O(h^3)
 \end{aligned}$$

This approximation has third ~~degree~~ ^{order} accuracy.

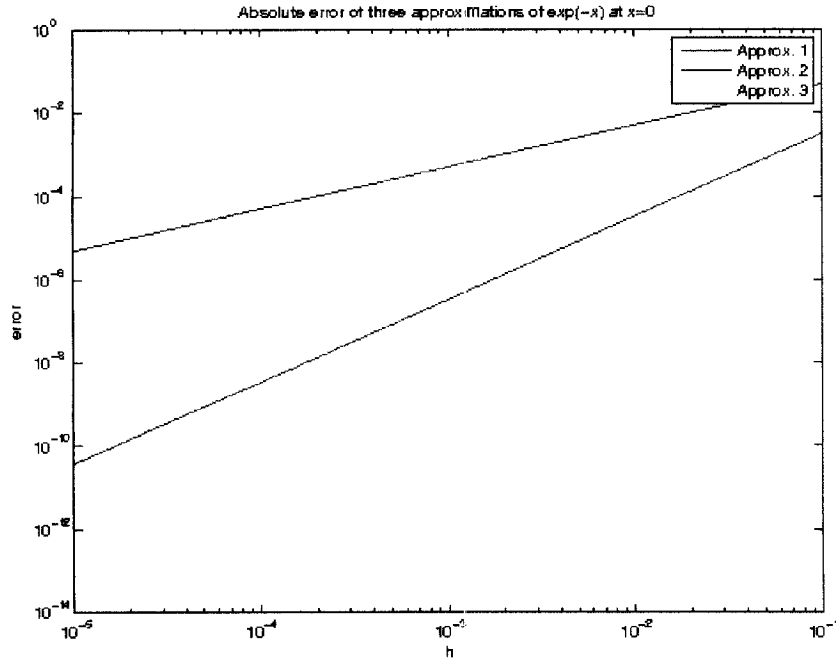
- (b) The values for the various approximations of $f'(x)$ evaluated at $x = 0$, where $f(x) = e^{-x}$, are shown below.

h	$f'(x)$ from i	$f'(x)$ from ii	$f'(x)$ from iii
10^{-1}	-0.9516258196404048	-0.9969054046707182	-1.0000801350225326
10^{-2}	-0.9950166250831893	-0.9999669155041468	-1.0000000830013770
10^{-3}	-0.9995001666249781	-0.9999996669165512	-1.000000000833786
10^{-4}	-0.9999500016666385	-0.999999966670003	-0.999999999991496
10^{-5}	-0.9999950000172396	-0.999999999621422	-1.000000000065510

The value of $f'(0)$ is known to be -1. The deviations from this value are listed below.

h	Error from i	Error from ii	Error from iii
10^{-1}	0.048374180359595176000	0.00309459532928180200	0.0000801350225325681
10^{-2}	0.004983374916810668000	0.00003308449585315287	8.3001377104707790e-8
10^{-3}	0.000499833375021907500	3.33083448822435460e-7	8.337863732776896e-11
10^{-4}	0.000049998333361500610	3.33299965404876270e-9	8.504308368628699e-13
10^{-5}	0.00000499982760378607	3.7857827983600600e-11	6.550981979103199e-12

(c)



(d) It shows from the table and graph above that the relative errors are following a pattern. We determined that the first approximation was linearly accurate. From the table, we see that the approximation is getting 10 times better as h gets 10 times smaller. Part (a) showed that the error should be about $\frac{1}{2}h$, which is just what we obtained. The second approximation was $O(h^2)$. On the table, we see that the approximation is getting 100 times better each time h gets 10 times smaller. The points on the graph show a trend that is roughly quadratic. Part (a) predicted the error term to be near $\frac{1}{3}h^2$. The third approximation for the first 3 terms are getting 1000 times better for every time h gets 10 times smaller, which was predicted by finding the order to be $O(h^3)$. These values were approximated by part (a) to be near $\frac{1}{12}h^3$. We can attribute the error in the 4th and 5th term to rounding error by the computer, but with better precision math, we could expect an error near 8.3×10^{-14} and 8.3×10^{-17} , respectively.

Go on!

3. Three point open Newton-Cotes formula

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(a) $\int_a^b f(x)dx \approx f(x_1)w_1 + f(x_2)w_2 + f(x_3)w_3$, where $x_j = jh + a$.

$$w_1 = \int_a^b \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} dx$$

$$w_2 = \int_a^b \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} dx$$

$$w_3 = \int_a^b \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} dx$$

For each w_j , choose a linear substitution to simplify the integral:

$$x = a + (z + 2)h,$$

$$x = a + (z + 1)h,$$

$$x = a + (z + 1)h,$$

where $h = (b - a)/4$ and each substitution is used with it's respective integral above. This change was made to force $z = 0$ to be a zero of the integrand for a simpler calculation. Evaluating, we obtain

$$w_1 = \int_{-2}^2 \frac{h}{2} z(z - 1) dz$$

$$w_2 = - \int_{-1}^3 h z(z - 2) dz$$

$$w_3 = \int_{-1}^3 h z(z - 1) dz$$

Thus, we have values $w_1 = \frac{8h}{3}$, $w_2 = \frac{-4h}{3}$, $w_3 = \frac{8h}{3}$, which yields

$$\int_a^b f(x) dx \approx \frac{4h}{3} (2f(x_1) - f(x_2) + 2f(x_3)).$$

(b) The values for the approximation of x^k over $[0, 1]$ are shown below.

k	Approximate	<u>Exact values</u>
0	1	?
1	1/2	.
2	1/3	
3	1/4	
4	148/768	

Thus the degree of precision is 3.

(c) The same substitutions are used as above to give

$$\int_a^b f(x) dx \approx \frac{4h}{3} \left(\frac{8}{5} f(x_1) + \frac{1}{3} f(x_2) + \frac{16}{15} f(x_3) \right).$$

The values for the approximation of x^k over $[0, 1]$ are shown below.

k	Approximate	<u>Exact</u>
0	1	?
1	1/2	.
2	1/3	
3	75/96	

For this approximation, the degree of precision is 2.

(d) The difference in precision comes from the same argument that gives Newton-Cotes approximations for even n and extra degree of precision. When we use the Weighted Mean Value Theorem to simplify the calculation of the error term, we find that the function being integrated has symmetry, which makes the the error term of degree $n + 1$ vanish. However, by moving the 3rd x -value over slightly, the symmetry is destroyed and with it the vanishing of the leading error term.