Midterm Exam

Math 128A

Fall 2008

Name Solutions

ID number ______________

Directions

- Do not begin until instructed to do so.
- You may use pencils, pens, erasers, calculators.
- Put away all cell phones or similar electronic devices.
- This exam is open book and open notes.
- Show all work for full credit.
- Keep your work as neat as possible. If we can't read it, we won't grade it.

Point totals

problem	1	2	3	4	5	6	total
score	/15	/10	/10	/10	/10	/10	/65

1. (5 points each, 15 total) Suppose that f is a C^∞ (all derivatives are continuous) function. To approximate f by either a polynomial or a piecewise polynomial we choose $n+1$ points at which our interpolant will match the function values. Determine if the statements are true or false. Write a short explanation of your answer.

   (a) If we approximate f by the n^{th} degree interpolating polynomial, then we can always get a better approximation by increasing the number of points.

   (b) The Lagrange form of the n^{th} degree interpolating polynomial is a better approximation than the Newton form.

   (c) If we approximate f by a clamped cubic spline, we can always get a better approximation by increasing the number of points.

a) This statement is false. Using higher degree polynomials can lead to oscillations in the interpolant that are not in the function being approximated. The error bound in thm 3.3 of the book depends on the behavior of the $n+1$st derivative & $\prod (x-x_i)$. This may not go to zero as n increases.

b) This Statement is false. The Lagrange form & Newton form are the same polynomial. ~~[illegible]~~

c) This statement is true. By thm 3.13 in the book, the error is bounded by a constant times the maximum spacing of the points to the fourth power. Therefore by using more points, the spacing decreases, and the bound on the error decreases.

2. (10 points) Find the Lagrange and Newton forms of the interpolating polynomial through the points (x,y) below.

x	1	2	4
y	6	8	16

Lagrange form

$$p(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)}6 + \frac{(x-1)(x-4)}{(2-1)(2-4)}8 + \frac{(x-1)(x-2)}{(4-1)(4-2)}16$$

$$p(x) = 2(x-2)(x-4) - 4(x-1)(x-4) + \frac{8}{3}(x-1)(x-2)$$

Newton form

$$p(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

$$f[x_0] = 6$$

$$f[x_0,x_1] = \frac{y_1-y_0}{x_1-x_0} = \frac{8-6}{2-1} = 2$$

$$f[x_1,x_2] = \frac{y_2-y_1}{x_2-x_1} = \frac{16-8}{4-2} = 4$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{4-2}{4-1} = \frac{2}{3}$$

$$p(x) = 6 + 2(x-1) + \frac{2}{3}(x-1)(x-2)$$

3. (a) (5 points) Expand $f(x+h)$ in a Taylor series about $h = 0$. Write out at least four terms of the series.

(b) (5 points) Use Taylor series to determine what derivative the formula below approximates. What is the order of accuracy of this approximation?

$$\frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$$

a)

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)$$

b)

$$\frac{f(x) - 2f(x+h) + f(x+2h)}{h^2} = \Big(f(x) - 2\big(f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + O(h^4)\big)$$

$$+ f(x) + 2hf'(x) + \frac{(2h)^2}{2} f''(x) + \frac{(2h)^3}{6} f'''(x) + O(h^4)\Big) \Big/ h^2$$

$$= h^{-2}\Big((f(x) - 2f(x) + f(x)) + h(-2f'(x) + 2f'(x)) + h^2\left(-\frac{2}{2} f''(x) + \frac{4}{2} f''(x)\right) + h^3\left(-\frac{2}{6} f'''(x) + \frac{8}{6} f'''(x)\right)$$

$$+ O(h^4)\Big)$$

$$= f''(x) + h f'''(x) + O(h^2)$$

This formula approximates the second derivative of f at x.

The error is $O(h)$, & so the approximation is first order accurate.

4. (10 points) Below is a cubic spline on the interval $x \in [1,3]$.

$$S(x) = \begin{cases} 1 + 3(x-1) - 2(x-1)^3 & \text{if } 1 \le x < 2 \\ 2 - 3(x-2) - 6(x-2)^2 - 2(x-2)^3 & \text{if } 2 \le x \le 3 \end{cases}$$

(a) Is S a natural spline? Explain your answer.

(b) Is S a not-a-knot spline? Explain your answer.

(a) A natural spline satisfies $S''=0$ at the endpoints.

$$S'(x) = \begin{cases} 3 - 6(x-1)^2 & 1 \le x < 2 \\ -3 - 12(x-2) - 6(x-2)^2 & 2 \le x \le 3 \end{cases}$$

$$S''(x) = \begin{cases} -12(x-1) & 1 \le x < 2 \\ -12 - 12(x-2) & 2 \le x \le 3 \end{cases}$$

$S''(1) = 0$

$S''(3) = -12 - 12(3-2) = -24 \neq 0$

Because $S''(3) \neq 0$, S is not a natural spline.

(b) S is a not-a-knot spline if S''' is continuous at $x=2$.

$$S'''(x) = \begin{cases} -12 & 1 \le x < 2 \\ -12 & 2 \le x \le 3 \end{cases}$$

Because $S'''(x)$ is continuous at $x=2$, S is a not-a-knot spline.

5. (10 points) Suppose we wish to find a polynomial that satisfies the conditions (a)–(d) below. There is a unique polynomial of degree at most n that satisfies these condtions. What is n? Explain your answer. Find the polynomial.

$$(a)\ p(0) = 0, \quad (b)\ p'(0) = 1, \quad (c)\ p''(0) = 1, \quad (d)\ p(1) = 0$$

$n = 3$. A 3^{rd} degree polynomial is defined by 4 coefficents. Conditions a–d give 4 equations, which can be used to solve for the 4 coefficents.

$$p(x) = a + bx + cx^2 + dx^3$$
$$p'(x) = b + 2cx + 3dx^2$$
$$p''(x) = 2c + 6dx$$

$$p(0) = a = 0$$
$$p'(0) = b = 1$$
$$p''(0) = 2c = 1$$
$$p(1) = a + b + c + d = 0$$

Solving this system,

$a = 0$, $b = 1$, $c = 1/2$, $d = -(a+b+c) = -3/2$

$$p(x) = x + \frac{1}{2}x^2 - \frac{3}{2}x^3$$

6. (10 points) Let $f \in C^2[x_0, x_1]$, and let P be the polynomial that interpolates f at x_0 and x_1. Show that for $x \in [x_0, x_1]$

$$|f(x) - P(x)| \leq \frac{1}{8} \max_{x \in [x_0, x_1]} |f''(x)|(x_1 - x_0)^2.$$

Because P interpolates f at two points, P is a 1^{st} degree polynomial.

By thm 3.3 in book

$$f(x) = P(x) + \frac{f''(\xi(x))}{2!}(x-x_0)(x-x_1)$$

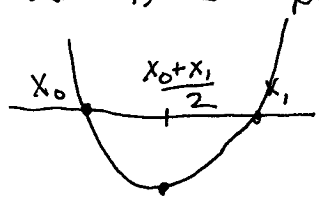
for some $\xi \in (x_0, x_1)$.

$$|f(x) - P(x)| = \left| \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1) \right|$$

$$\leq \max_{x \in [x_0, x_1]} \left| \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1) \right|$$

$$|f''(\xi(x))| \leq \max_{x \in [x_0, x_1]} |f''(x)|.$$

$(x-x_0)(x-x_1)$ is a parabola with zeros at $x = x_0$ & $x = x_1$.



The max of $|(x-x_0)(x-x_1)|$ is attained at $x = \frac{x_0+x_1}{2}$.

$$\max_{x \in [x_0, x_1]} |(x-x_0)(x-x_1)| = \left| \left(\frac{x_0+x_1}{2} - x_0\right)\left(\frac{x_0+x_1}{2} - x_1\right) \right| = \frac{1}{4}(x_1 - x_0)^2$$

Therefore

$$|f(x) - P(x)| \leq \max_{x \in [x_0, x_1]} \left| \frac{f''(\xi)}{2}(x-x_0)(x-x_1) \right| = \frac{1}{8} \max_{x \in [x_0, x_1]} |f''(x)| \, (x_1 - x_0)^2.$$