Math 228A
Homework 1
Due Tuesday, 10/12/10

1. Let L be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.

   (a) Find the eigenfunctions and corresponding eigenvalues of L.

   (b) Show that the eigenfunctions are orthogonal in the $L^2[0,1]$ inner product:
   $$\langle u, v\rangle = \int_0^1 uv\,dx.$$

   (c) It can be shown that the eigenfunctions, $\phi_j(x)$, form a complete set in $L^2[0,1]$. This means that for any $f \in L^2[0,1]$, $f(x) = \sum_j \alpha_j \phi_j(x)$. Express the solution to
   $$u_{xx} = f, \quad u_x(0) = u_x(1) = 0, \tag{1}$$
   as a series solution of the eigenfunctions.

   (d) Note that equation (1) does not have a solution for all f. Express the condition for existence of a solution in terms of the eigenfunctions of L.

2. Define the functional $F : X \to \Re$ by
   $$F(u) = \int_0^1 \frac{1}{2}(u_x)^2 + fu\,dx,$$
   where X is the space of real valued functions on $[0,1]$ that have at least one continuous derivative and are zero at $x = 0$ and $x = 1$. The Frechet derivative of F at a point u is defined to be the linear operator $F'(u)$ for which
   $$F(u+v) = F(u) + F'(u)v + R(v),$$
   where
   $$\lim_{||v||\to 0} \frac{||R(v)||}{||v||} = 0.$$
   One way to compute the derivative is
   $$F'(u)v = \lim_{\epsilon\to 0} \frac{F(u+\epsilon v) - F(u)}{\epsilon}.$$
   Note that this looks just like a directional derivative.

   (a) Compute the Frechet derivative of F.

   (b) $u \in X$ is a critical point of F if $F'(u)v = 0$ for all $v \in X$. Show that if u is a solution to the Poisson equation
   $$u_{xx} = f, \quad u(0) = u(1) = 0,$$
   then it is a critical point of F.

   Finite element methods are based on these "weak formulations" of the problem. The Ritz method is based on minimizing F and the Galerkin method is based on finding the critical points of $F'(u)$.

3. (a) Using a Taylor expansion, derive the finite difference formula to approximate the second derivative at x using function values at $x - h/2$, x, and $x + h$. How accurate is the finite difference approximation?

   (b) Perform a refinement study to test the accuracy of the formula you derived by approximating $u''(1)$, where $u(x) = \cos(2\pi x)$.

   (c) Does the problem above provide a good test of the accuracy? Explain why or why not. If not, perform another test.

   (d) Derive an expression for the quadratic polynomial that interpolates the data $(x - h/2, u(x - h/2))$, $(x, u(x))$, $(x + h, u(x + h))$. How is the finite difference formula your derived in problem 3a related to the interpolating polynomial?