

Math 228A

Homework 1

Due Thursday, 10/16/08, 1:40 p.m.

1. Let L be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.

(a) Find the eigenfunctions and corresponding eigenvalues of L .

(b) Show that the eigenfunctions are orthogonal in the $L^2[0, 1]$ inner product:

$$\langle u, v \rangle = \int_0^1 uv \, dx.$$

(c) It can be shown that the eigenfunctions, $\phi_j(x)$, form a complete set in $L^2[0, 1]$. This means that for any $f \in L^2[0, 1]$, $f(x) = \sum_j \alpha_j \phi_j(x)$. Express the solution to

$$u_{xx} = f, \quad u_x(0) = u_x(1) = 0, \tag{1}$$

as a series solution of the eigenfunctions.

(d) Note that equation (1) does not have a solution for all f . Express the condition for existence of a solution in terms of the eigenfunctions of L .

2. Define the functional $F : X \rightarrow \Re$ by

$$F(u) = \int_0^1 \frac{1}{2} (u_x)^2 + fu \, dx,$$

where X is the space of real valued functions on $[0, 1]$ that have at least one continuous derivative and are zero at $x = 0$ and $x = 1$. The Frechet derivative of F at a point u is defined to be the linear operator $F'(u)$ for which

$$F(u + v) = F(u) + F'(u)v + R(v),$$

where

$$\lim_{\|v\| \rightarrow 0} \frac{\|R(v)\|}{\|v\|} = 0.$$

One way to compute the derivative is

$$F'(u)v = \lim_{\epsilon \rightarrow 0} \frac{F(u + \epsilon v) - F(u)}{\epsilon}.$$

Note that this looks just like a directional derivative.

(a) Compute the Frechet derivative of F .

(b) $u \in X$ is a critical point of F if $F'(u)v = 0$ for all $v \in X$. Show that if u is a solution to the Poisson equation

$$u_{xx} = f, \quad u(0) = u(1) = 0,$$

then it is a critical point of F .

Finite element methods are based on these “weak formulations” of the problem. The Ritz method is based on minimizing F and the Galerkin method is based on finding the critical points of $F'(u)$.

3. Consider the finite difference formula

$$\frac{u(x - 2h) - 4u(x - h) + 3u(x)}{2h}.$$

- (a) Using a Taylor series, determine what derivative this formula approximates.
 - (b) What is the error of the approximation?
 - (c) Derive an expression for the quadratic polynomial that interpolates the data $(x - 2h, u(x - 2h))$, $(x - h, u(x - h))$, $(x, u(x))$. How is the finite difference related to interpolating polynomial?
4. Use the standard 3-point discretization of the Laplacian on a regular mesh to find a numerical solution to the PDE below. Perform a refinement study using the exact solution to compute the error that shows the rate of convergence for both the 1-norm and the max norm.

$$u_{xx} = \exp(x), \quad u(0) = 0, \quad u(1) = 1.$$

5. We have typically discretized the interval $[0, 1]$ into equally spaced points $x_j = jh$ for $j = 0 \dots N + 1$ with $h = 1/(N + 1)$. Another common discretization is the *cell centered* mesh, in which $[0, 1]$ is discretized into N cells. The grid points are placed at centers of the cells: $x_j = (j - 1/2)h$ for $j = 1 \dots N$ where $h = 1/N$. This type of discretization is more natural for some problems, particularly those with Neumann boundary conditions.

- (a) We may write $u_{xx} = (-J)_x$, where $J = -u_x$ is the diffusive flux. Suppose we discretize this problem by using a centered difference to compute the flux at the cell edges, $J_{j-1/2}$, followed by another centered difference of the flux. Show that at interior points this gives the standard second-order discretization of u_{xx} .
- (b) Again using the idea of flux differencing, derive the discrete approximation to u_{xx} at the first interior grid point adjacent to a boundary with Neumann boundary condition $u_x = g$.
- (c) The cell-centered discretization seems awkward for Dirichlet boundary conditions because the spacing from the boundary to the first interior grid point is not equal to the spacing between interior grid points. Derive a finite difference approximation for u_{xx} at the grid points adjacent to the boundary involving the boundary value and the first two interior points. What is the local truncation error of your approximation?
- (d) Another way think about discretizing u_{xx} near the boundary is to imagine a *ghost cell* outside the domain which contains a function value, and then the same discrete operator can be applied at first interior point as is applied at interior points. For example, we may set the ghost value, u_0 at $x_0 = -h/2$ by linear extrapolation using the boundary value and the first interior point: $u_0 = 2u(0) - u_1$. What is the resulting finite difference formula for u_{xx} at the first interior point? What is the local truncation error at these points?

- (e) Use a cell-centered mesh to find a numerical solution to the PDE below. Perform refinement studies to determine the rate of convergence for the two different discretization of the Dirichlet boundary condition from parts 5c and 5d.

$$u_{xx} = \exp(-10(x - 0.3)^2), \quad u(0) = 1, \quad u_x(1) = 0.1$$

6. As a general rule, we usually think that an $O(h^p)$ local truncation error (LTE) leads to an $O(h^p)$ error. However, in some cases the LTE can be lower order at some points without lowering the order of the error. Consider the standard second-order discretization of the Poisson equation on $[0, 1]$ with homogeneous boundary conditions. The standard discretization of this problem gives an $O(h^2)$ LTE provided the the solution is at least C^4 . The LTE may be lower order because the solution is not C^4 or because we use a lower order discretization (see problem 5) at some points.

- (a) What equation relates the error and the LTE?
- (b) Suppose that the LTE is $O(h^p)$ at the first interior point ($x_1 = h$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error? Hint: Use equation (2.46) from LeVeque to aid in your argument.
- (c) Suppose that the LTE is $O(h^p)$ at an interior point (i.e. a point that does not limit to the boundary as $h \rightarrow 0$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error?
- (d) Verify the results of your analysis from parts (b) and (c) using numerical tests.