Math 228A
Homework 2
Due Thursday, 11/13/08, 1:40 p.m.

1. Use Jacobi, Gauss-Seidel, and SOR (with optimal ω) to solve

$$\Delta u = -\exp\left(-(x-0.25)^2 - (y-0.6)^2\right)$$

   on the unit square $(0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions. Find the solution for mesh spacings of $h = 2^{-5}$, 2^{-6}, and 2^{-7}. What tolerance did you use? What stopping criteria did you use? What value of ω did you use? Report the number of iterations it took to reach convergence for each method for each mesh.

2. The convergence rate of SOR is very sensitive to the choice of ω near $\omega = \omega_{opt}$. In this problem, you will numerically estimate the spectral radius of the SOR iteration matrix as a function of ω. Let $e^{(0)}$ be the initial error, and $e^{(k)}$ be the error after k steps of the iteration. The spectral radius of the iteration matrix can be estimated by

$$\rho \approx \left(\frac{\|e^{(k)}\|_\infty}{\|e^{(0)}\|_\infty} \right)^{1/k}$$

   for large k. For some fixed mesh size, make a plot of the spectral radius of the SOR matrix, ρ_ω, as a function of ω for $0 \leq \omega \leq 2$. An easy test problem to use is $\Delta u = 0$ with homogeneous Dirichlet boundary conditions. Because the solution is $u = 0$, $e^{(k)} = u^{(k)}$. Comment on your results.

   Note that this is an inefficient way of estimating ω_{opt}. There are more efficient methods for estimating ω_{opt} in practice.

3. When solving parabolic equations numerically, one frequently needs to solve an equation of the form

$$u - \delta \Delta u = f,$$

   where $\delta > 0$. The analysis and numerical methods we have discussed for the Poisson equation can be applied to the above equation. Suppose we are solving the above equation on the unit square with Dirichlet boundary conditions. Use the standard five point stencil for the discrete Laplacian.

   (a) Write the discretized equation at grid point (i,j), and give an expression for the Jacobi iteration at a point. i.e. $u_{i,j}^{k+1} = \ldots$

   (b) Analytically compute the eigenvalues of the Jacobi iteration matrix, and show that the Jacobi iteration converges.

   (c) If $h = 10^{-2}$ and $\delta = 10^{-3}$, how many iterations of Jacobi would it take to reduce the error by a factor of 10^{-6}? How many iterations would it take for the Poisson equation?

   (d) Repeat the previous part for SOR using that the spectral radius of SOR is

$$\rho_{\text{sor}} = \omega_{\text{opt}} - 1,$$

where

$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\rho_J^2}},$$

and where ρ_J is the spectral radius of Jacobi.

4. Periodic boundary conditions for the one dimensional Poisson equation on $(0,1)$ are $u(0) = u(1)$ and $u_x(0) = u_x(1)$. These boundary conditions are easy to discretize, but lead to a singular system to solve. For example, using the standard discretization, $x_j = jh$ where $h = 1/(N+1)$, the discrete Laplacian at x_0 is $h^{-2}(u_N - 2u_0 + u_1)$.

   (a) Write the discrete Laplacian for periodic boundary conditions in one dimension as a matrix. Show that this matrix is singular, and find the vectors that span the null space. (Note that this matrix is symmetric, and so you have found the null space of the adjoint).

   (b) What is the discrete solvability condition for the discretized Poisson equation with periodic boundary conditions in one dimension? What is the discrete solvability condition in two dimensions?

   (c) Show that v is in the null space of the matrix A, if and only if v is an eigenvector of the iteration matrix $T = M^{-1}N$ with eigenvalue 1, where $A = M - N$. The iteration will converge if the discrete solvability condition is satisfied provided the other eigenvalues are less than 1 in magnitude (true for Gauss-Seidel, but not for Jacobi).

   (d) Use Gauss-Seidel to solve the two dimensional Poisson equation below with periodic boundary conditions in both the x and y directions.

$$u_{xx} + u_{yy} = -4\pi^2 \mathrm{e}^{\cos(2\pi x)} \cos(2\pi x)\big(1 + \cos(2\pi x)\big) \sin(2\pi y)$$

   To make the solution unique, find the solution with mean value 0. Make sure that you enforce the discrete solvability condition before iterating. Verify that you found the correct solution by comparing with the exact solution $u = \exp(\cos(2\pi x))\sin(2\pi y)$.

   Implementation Suggestion: An easy way to implement iterative solvers on periodic problems is by using ghost cells. Define the unknowns at the black dots (see below), but before performing an iteration, set the open dots using periodicity.