Math 228A
Homework 2
Due Friday, 10/22/08, 4:00

1. Use the standard 3-point discretization of the Laplacian on a regular mesh to find a numerical solution to the PDEs below. Perform a refinement study using the exact solution to compute the error that shows the rate of convergence for both the 1-norm and the max norm.

   (a) $u_{xx} = \exp(x), \quad u(0) = 0, \quad u(1) = 1$

   (b) $u_{xx} = 2\cos^2(\pi x), \quad u_x(0) = 0, \quad u_x(1) = 1$

2. Propose a discretization scheme for

$$u_{xx} = f, \quad u_x(0) - \alpha u(0) = g, \quad u(1) = b.$$

   What is the form of the matrix and right hand side in your discrete equations? What order of accuracy do you expect?

3. As a general rule, we usually think that an $O(h^p)$ local truncation error (LTE) leads to an $O(h^p)$ error. However, in some cases the LTE can be lower order at some points without lowering the order of the error. Consider the standard second-order discretization of the Poisson equation on $[0, 1]$ with homogeneous boundary conditions. The standard discretization of this problem gives an $O(h^2)$ LTE provided the the solution is at least C^4. The LTE may be lower order because the solution is not C^4 or because we use a lower order discretization at some points.

   (a) Suppose that the LTE is $O(h^p)$ at the first grid point ($x_1 = h$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error? Hint: Use equation (2.46) from LeVeque to aid in your argument.

   (b) Suppose that the LTE is $O(h^p)$ at an interior point (i.e. a point that does not limit to the boundary as $h \to 0$). What effect does this have on the error? What is the smallest value of p that gives a second order accurate error?

   (c) Verify the results of your analysis from parts (a) and (b) using numerical tests.