Math 228A
Homework 4
Due Thursday, 12/11/08, 5:00 P.M.

1. Work out reasonable choices for a , b, and c in the approximate Cholesky factorization described below.

2. Write a program to solve the discrete Poisson equation on the unit square using preconditioned conjugate gradient. Set up a test problem and compare the number of iterations and efficiency of using (i) no preconditioning, (ii) approximate Cholesky preconditioning, and (iii) SSOR preconditioning. Run your tests for different grid sizes. How does the number of iterations increase as the grid is refined?

   These preconditioners are described below. In both cases the necessary matrix solve is accomplished by forward substitution followed by back substitution. Note that there are two typos in the PCG algorithm in our textbook. See your class notes, another textbook, or the author's webpage for the correct algorithm.

   Approximate Cholesky preconditioning The preconditioner is of the form $M = LL^{\mathrm{T}}$ where L is lower triangular. One way to choose L is by assuming that it has the same sparsity pattern as the original matrix A. For the discrete Poisson equation in stencil notation, L is of the form

$$L = \begin{bmatrix} & & \\ b & a & \\ & c & \end{bmatrix}$$

   Choose a, b, and c so that $LL^{\mathrm{T}} \approx A$. For example a, b, and c may be chosen so that the stencil of LL^{T} matches the stencil of A in as many entries as possible. A better way to pick a, b, and c is by examining the Taylor expansion of $LL^{\mathrm{T}}\boldsymbol{u}$, and choosing a, b, and c to give the best approximation to the Laplacian (e.g. try $b = c = -a/2$).

   SSOR preconditioning Symmetric SOR (SSOR) consists of one forward sweep of SOR followed by one backward sweep of SOR. For the discrete Poisson equation, one step of SSOR is

$$\begin{aligned} u_{i,j}^{k+1/2} &= \frac{\omega}{4}(u_{i-1,j}^{k+1/2} + u_{i,j-1}^{k+1/2} + u_{i+1,j}^{k} + u_{i,j+1}^{k} - h^2 f_{i,j}) + (1-\omega)u_{i,j}^{k} \\ u_{i,j}^{k+1} &= \frac{\omega}{4}(u_{i-1,j}^{k+1/2} + u_{i,j-1}^{k+1/2} + u_{i+1,j}^{k+1} + u_{i,j+1}^{k+1} - h^2 f_{i,j}) + (1-\omega)u_{i,j}^{k+1/2}. \end{aligned}$$

   It can be shown that one step of SSOR in matrix form (for natural row ordering) is equivalent to

$$\frac{1}{\omega(2-\omega)}(D - \omega L)D^{-1}(D - \omega U)(\boldsymbol{u}^{k+1} - \boldsymbol{u}^{k}) = \boldsymbol{f},$$

   where $A = D - L - U$.

   For the constant coefficient problem, this suggests the preconditioner.

$$M = (D - \omega L)(D - \omega U).$$